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# GEOMETRIC EXERCISES FOR ALGEBRAIC SOLUTION

SECOND-YEAR MATHEMATICS
FOR SECONDARY SCHOOLS

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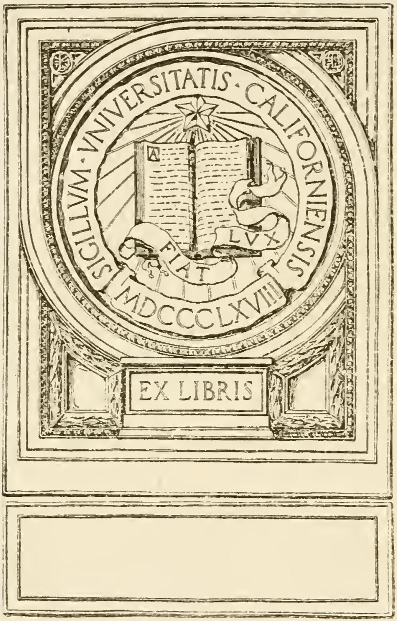
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# GEOMETRIC EXERCISES FOR ALGEBRAIC SOLUTION

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# Geometric Exercises for Algebraic Solution

## SECOND-YEAR MATHEMATICS FOR SECONDARY SCHOOLS

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School of Education Manuals
Secondary Texts

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# PREFACE

The reasons against the plan in common vogue in secondary schools of breaking the continuity of algebra by dropping it for a whole year after barely starting it, are numerous and strong. Perhaps the only argument that need be cited here is that this procedure creates—and that, too, gratuitously—an educational gap, and that the modern science of education is fundamentally hostile to gaps. Everywhere today interest and continuity in educational procedure are insisted upon. The hiatus here is no less glaring than that between the grades and the first high-school year. The fact that the former is less talked about than the now "classical gap" means only that fewer students of education are interesting themselves in the virtues of mathematical than of general education.

With no other subject of the curriculum does a loss of continuity and consecutiveness work so great havoc as with mathematics. To attain high educational results from any body of mathematical truths, once grasped, it is profoundly important that subsequent work be so planned and executed as to lead the learner to see their value and to feel their power through manifold uses. They must be followed up consecutively, and before very long. The view, here and there heard, that it is best for the learner to let previously acquired knowledge and skill get entirely away from him, that he may return to it with renewed vigor and pleasure, is based on the assumption that former teaching has either killed the interest, or never generated any interest, in the subject and, hence, the view is without grounds to support it. Mathematical education has suffered too long, and has even yet found too little relief, from the conception of the drill-master, rather than the teacher.

The University High School offers an exceptionally rich curriculum of secondary studies and, under the spur of the vigorous industrial and commercial community, whose interests it aspires both to subserve and to focus, any loss of time to the pupil, due to a loosely articulated sequence of studies, comes home to the teachers with peculiar force. Thus it came about that the teachers of mathematics of this school felt constrained to set about to close the breach in algebra. It was their avowed purpose to devise some practicable means of enabling mathematical pupils during the second year, at least to hold the algebraic ground made during the first year, and, perhaps, to add a little new territory.

With a separate text in both algebra and geometry, the German plan of carrying the subjects along abreast was tried with indifferent success. Teachers were new to the plan, pupils found it distracting and distasteful to pass back and forth at short intervals from one subject to the other, with no good apparent reason, and American texts are ill-adapted to the plan. It soon became evident that teachers must, to preserve the unity of geometry, prepare a set of algebraic exercises and problems that would call for *geometrical thinking* and for *algebraic technique*. The exercises were to be so chosen as to cover algebra to and through quadratics.

This little manual is the outcome of their work. A thoroughly correlated body of mathematical truth, organized about a geometrical center is what is wanted. This list of supplementary exercises is a step in the direction of such correlation. It has been and will be used to supplement the work in second-year geometry in The University High School. To make this supplementary use of the exercises practicable is the prime purpose of publishing this manual. It is hoped that in the printed form it may also be found helpful to other teachers as a thesaurus of interesting supplementary material to either algebra or geometry in the second high-school year, if the other subject is studied in the first year.

The University High School teachers assign selected exercises from this manual as a part of the pupils' home work. These exercises subserve the office of "originals" and applications for geometry, and of reviews and applications for algebra. The importance of this twofold function is felt in the saving of time that results.

Acknowledgments are due the officials of The University High School for the privilege of using classes of that institution as a sort of experimental laboratory in working out the larger plan of which this manual is a part. To Dean W. B. Owen, particularly, is due our very grateful appreciation for all sympathy and every encouragement.

That this little manual may lighten in some degree the burdens of the duty of collating problems to which every mathematical teacher, who strives to keep awake an interest in his subject, is enjoined by the demands of modern mathematical pedagogy, is the earnest wish of its compilers!

THE AUTHORS

CHICAGO
AUGUST, 1907

## GEOMETRIC EXERCISES FOR ALGEBRAIC SOLUTION

### § 1. On Showing the Constitution of Angles Algebraically

1. An angle of $60^\circ$ is made by doubling an angle, $x$; or the angle, $60^\circ$, may be denoted by $2x$. What is the value of $x$?

2. An angle of $84^\circ$ is made by adding together twelve equal angles, $a$; what is the value in degrees of $a$? Suggestion: $12a = 84^\circ$; find $a$.

3. The angle $95^\circ$ is made by adding $35^\circ$ to the sum of six equal angles, $x$. Find $x$ in degrees. Suggestion: $6x + 35^\circ = 95^\circ$. Solve for $x$.

4. The angle $48^\circ$ is made as shown by this expression: $4x + 8^\circ$. Find the constituent angle, $x$.

5. The angle $75^\circ$ is constituted as shown by $8x - 5^\circ$. Find $x$.

6. The angle $89^\circ$ is composed thus: $25y - 11^\circ$. Find $y$.

7. The angle $73^\circ$ is composed thus: $5z + 3^\circ$. Find $z$.

8. Find $y$ in each of the following cases, supposing the angle $90^\circ$ is composed thus:

(1) $9y$;
(2) $18y$;
(3) $45y$;
(4) $\frac{y}{2}$;
(5) $\frac{y}{5}$;
(6) $2y + 10^\circ$;
(7) $10y + 10^\circ$;
(8) $2y - 10^\circ$;
(9) $10y - 10^\circ$;
(10) $7y + 20^\circ$;
(11) $11y - 20^\circ$;
(12) $13y + 12^\circ$;
(13) $14y + 20^\circ$;
(14) $y + 25^\circ$;
(15) $40y - 30^\circ$.

9. Find $y$ supposing the angle $80^\circ$ to be composed as shown by each of the 15 expressions of Problem 8.

10. Find $z$, in each case, supposing the angle $360^\circ$ is composed thus:

(1) $36z$;
(2) $20z$;
(3) $18z$;
(4) $3z + 60^\circ$;
(5) $30z + 60^\circ$;
(6) $40z - 40^\circ$;
(7) $10z + 40^\circ$;
(8) $80z - 40^\circ$;
(9) $9z - 18^\circ$;
(10) $5z + 35^\circ$;
(11) $18z - 36^\circ$;
(12) $90z + 180^\circ$;
(13) $9z + 27^\circ$;
(14) $az - 18^\circ$;
(15) $cz + d^\circ$.

11. Find $z$, in each case, supposing the angle 270° to be composed as the expressions in Problem 10 show.

REMARK.—The *numerical measures* of angles, i. e., the abstract numbers that show how many units of angular magnitude the angle in question contains, may be made up of abstract algebraic numbers.

12. Find the number $x$ in each case, supposing the numbers in the right members of the equations to be the *numerical measures* of angles composed out of $x$ as shown by the algebraic expressions on the left:

(1) $x^2 = 144$;
(2) $x^2 + 2x = 48$;
(3) $x^2 + 4x = 32$;
(4) $x^2 - 2x = 48$;
(5) $18x + 3x^2 = 120$;
(6) $3x^2 - 18x = 120$;
(7) $20x + 2x^2 = 238$;
(8) $9x + 3x^2 = 30$;
(9) $9x^2 - 27x = 90$;
(10) $5x^2 + 15x = 140$;
(11) $6x + 2x^2 = 56$;
(12) $3x^2 - 42x = 45$;
(13) $x^2 - 14x = 72$;
(14) $2x^2 + 16x = 210$;
(15) $3x^2 - 3x = 270$;
(16) $4x^2 - 24x = 220$.

## §2. Of Complementary Angles

1. The angles $x$ and 38° are complementary. What is the value of $x$?

2. How many degrees are there in an angle that is the complement of 2 times itself? Of 3 times itself? Of 5 times itself? Of 12 times itself? Of $a$ times itself?

3. The difference of two complementary angles is 36°. Find them.

4. Find two complementary angles whose difference is 10°; 24°; 47°; 22½°; $a$°.

5. $x + 20°$ is the complement of $4x + 10°$. Find the angle $x$. Find $x + 20°$. Find $4x + 10°$.

6. How many degrees are there in an angle whose complement is ¾ of a right angle?

7. The difference between an angle and its complement is 5°. Find the angle.

8. State, by an equation, that $x$ and $y$ are complementary angles, and solve the equation for $x$. For $y$.

9. $x$ and $y$ are complementary angles, and their difference is 36°. Find $x$ and $y$.

10. $ax$ and $bx$ are complementary angles; find the value of $x$.

11. $\frac{x}{a}$ and $\frac{x}{b}$ are complementary angles; find $x$.

12. $(a+b)x$ and $(a-b)x$ are complementary angles. Find the angle $x$.

13. $\frac{y}{a+b}$ and $\frac{y}{a-b}$ are complementary angles. Find the angle $y$. Find $\frac{y}{a+b}$.

14. $4x$ and $-2x$ are complementary angles. Find the angle $x$. Find $-2x$.

15. $3x$ and 120° are complementary angles. Find $x$. Find $3x$.

16. $y$ and 125° are complementary angles. Find $y$.

17. $\frac{x}{a}$ and $\frac{y}{b}$ are complementary angles and their difference is 42°. Find the angles $x$ and $y$.

18. Find each of two complementary angles that are in the ratio of 1 : 2; of 1 : 3; of 2 : 5; of 4 : 5; of $a : b$.

19. 90° is sometimes expressed as $\frac{\pi}{2}$ radians. $x$ and $\frac{\pi}{4}$ are complementary. Find $x$ in terms of $\pi$.

20. If 6° be taken from one of two complementary angles and added to the other, the ratio of the two angles, thus found, is 2 : 7. Find the angles.

21. One of two complementary angles is diminished by 15° and then quadrupled. The other is increased by 10° and trebled. The difference of the angles, thus obtained, is 95°. Find the two complementary angles.

22. $x+5$ degrees and $y+10$ degrees are complementary and their ratio is $3:5$. Find $x$ and $y$.

23. $x+18$ degrees and $y-36$ degrees are complementary, and $x:y=3:1$. Find $x$ and $y$.

## §3. Of Supplementary Angles

1. The angles $x$ and 78° are supplementary. What is the value of $x$?

2. How many degrees are there in an angle that is the supplement of 3 times itself? Of 5 times itself? Of 8 times itself? Of 19 times itself? Of $a$ times itself?

3. The difference of two supplementary angles is 80°. Find them.

4. Find two supplementary angles whose difference is 20°; 48°; $67\frac{1}{2}°$; $d°$.

5. $x+100$ degrees and $9x-20$ degrees are supplementary angles. Find the angle $x$.

6. $2x+30°$ is the supplement of $6x+10°$. Find the angle $x$; $2x+30°$; $6x+10°$.

7. How many degrees are there in an angle whose supplement is $\frac{3}{5}$ of a right angle? $\frac{17}{45}$ of a right angle? $\frac{a}{b}$ of a right angle?

8. State, by an equation, that $x$ and $y$ are supplementary angles and solve for $x$; for $y$.

9. $x$ and $y$ are supplementary angles and their difference is 66°. Find $x$ and $y$.

10. $cx$ and $dx$ are supplementary angles. Find $x$. Find $cx$. Find $dx$.

11. $\frac{x}{c}$ and $\frac{x}{d}$ are supplementary angles. Find the value of $x$. Of $\frac{x}{c}$. Of $\frac{x}{d}$.

12. $\frac{x}{c+d}$ and $\frac{x}{c-d}$ are supplementary angles. Find $x$. Find $\frac{x}{c+d}$. Find $\frac{x}{c-d}$.

13. $7x$ and $-3x$ are supplementary angles. Find $x$. Find $-3x$.

14. $7x$ and 201° are supplementary angles. Find $x$. Find $7x$.

15. $16x$ and $-6x$ are supplementary angles. Find them.

16. $\frac{x}{a}$ and $\frac{y}{b}$ are supplementary angles and their difference is 74°. Find $\frac{x}{a}$ and $\frac{y}{b}$. Find $x$ and $y$.

17. Find each of two supplementary angles that are in the ratio to each other of $2:5$; $3:7$; $m:n$; $a+b$ to $a-b$.

18. The angles $3x+y$ and $x-y$ are supplementary and their difference is 72°. Find the angles $x$ and $y$.

19. If 11° be added to one of two supplementary angles and 16° be taken away from the other, the two angles, so obtained, will be to each other as $3:4$. Find the two supplementary angles.

20. If 15° be subtracted from 3 times one of two supplementary angles and 12° be added to 5 times the other, the difference of the two angles, thus formed, will be 33°. Find the two supplementary angles.

21. The ratio of 2 times one of two supplementary angles to 8 times the other is as $1:2$. Find the two supplementary angles.

22. 180° is sometimes expressed as $\pi$ radians. $y$ and $\frac{3}{4}\pi$ are supplementary angles. Express $y$ in terms of $\pi$.

23. $2x+10°$ and $3y-5°$ are supplementary angles and $x:y=7:2$. Find $x$ and $y$.

## §4. Of Complementary and Supplementary Angles

1. $x$ and $y$ are complementary and $10x$ and $y$ are supplementary angles. Find $x$ and $y$.

2. $4x$ and $2y$ are complementary and $5x$ and $-2y$ are supplementary angles. Find $x$ and $y$.

3. How many degrees are there in an angle whose complement and supplement together are 210°? 150°? 110°? 135°? −130°? $d$°?

4. How many degrees are there in an angle whose supplement is 3 times its complement? 4 times its complement? 2 times its complement? $n$ times its complement?

5. How many degrees are there in an angle whose complement is to its supplement as $1:3$? $3:7$? $5:8$? $16:25$? $1:2$? $2:3$? $m:n$?

6. Find an angle whose supplement increased by 30° is $\frac{5}{2}$ of its complement.

7. Find an angle whose supplement increased by 10° is 6 times its complement.

8. Find the angle whose supplement divided by 4 is $\frac{4}{7}$ of its complement.

9. How many degrees are there in an angle whose supplement, increased by 48°, is to its complement, diminished by 12°, as $7:2$?

10. Find an angle whose complement and supplement are in the ratio of $5:14$; $a:b$.

11. The complement of $a$ is $\frac{1}{5}$ of the supplement of $a$. Find $a$.

12. $x$ and $y$ are supplementary, and $x$ and $-y$ are complementary angles. Find $x$ and $y$.

13. $x$ and $y$ are supplementary, and $3x$ and $-2y$ are complementary angles. Find $x$ and $y$.

14. $4x$ and $3y$ are supplementary, and $2x$ and $2y$ are complementary angles. Find $x$ and $y$.

15. $3y$ is the supplement of $4x$ and the complement of $2x$. Find $x$ and $y$.

16. Under the following sets of conditions find, for each set, the angles $x$ and $y$:

| | Supplementary Pairs | | Complementary Pairs | |
|---|---|---|---|---|
| (1) | $3x$ | and $6y$, | and $2x$ | and $-2y$; |
| (2) | $2x$ | and $y$, | and $x$ | and $-\frac{y}{2}$; |
| (3) | $4x$ | and $-2y$, | and $x$ | and $\frac{2y}{5}$; |
| (4) | $3x$ | and $y$, | and $2x$ | and $-y$; |
| (5) | $6x$ | and $-5y$, | and $2x$ | and $-y$; |
| (6) | $4x$ | and $3y$, | and $7x$ | and $-6y$; |
| (7) | $6x$ | and $9y$, | and $8x$ | and $-18y$; |
| (8) | $5x$ | and $5y$, | and $8x$ | and $-\frac{y}{4}$; |
| (9) | $x-4$ | and $y+3$, | and $2(x-4)$ | and $-\frac{y+3}{4}$; |
| (10) | $\frac{x+8}{2}$ | and $\frac{y-10}{3}$, | and $\frac{x+8}{4}$ | and $\frac{y-50}{5}$; |
| (11) | $\frac{3}{4}(x+y)$ | and $\frac{5}{3}(x-y)$, | and $\frac{7}{16}(x+y)$ | and $\frac{5}{9}(x-y)$; |
| (12) | $ax$ | and $by$, | and $ax$ | and $-2by$; |
| (13) | $\frac{x}{a}$ | and $\frac{y}{b}$, | and $\frac{x}{a}$ | and $-\frac{y}{2b}$; |
| (14) | $3ax$ | and $2by$, | and $ax$ | and $\frac{8by}{3}$. |

17. $x$'s complement is $\frac{1}{6}$ of $x$'s supplement. Find the angle, $x$.

## §5. Of Other Angles Made by Intersecting Lines

1. A straight line, $AB$, is met at $C$, between $A$ and $B$, by another straight line, $CD$, making two supplementary adjacent angles that are in the ratio 1 : 2. Find them.

2. Find both of two supplementary adjacent angles that are in the ratio of $2:3$; of $2:5$; of $3:7$; of $5:9$; of $m:n$.

3. A straight line, $AB$, is met at $C$, between $A$ and $B$, by two straight lines making angles on one side of $AB$ that are in the ratio $1:2:3$. Find their values in degrees.

4. Solve a problem similar to that in Problem 3 if the angles are in the ratio $1:3:4$; $1:4:7$; $2:5:8$; $2:7:9$; $\frac{1}{2}:\frac{3}{4}:1$; $\frac{1}{2}:\frac{3}{4}:\frac{5}{6}$; $a:b:c$.

5. If three straight lines meet the line, $AB$, of Problem 3 at $C$, giving four angles on one side of $AB$ that are in the ratio of $1:2:3:4$; find the angles.

6. Find the four angles, described in Problem 5, if their ratios are $1:2:5:9$; $2:5:6:7$; $\frac{1}{4}:\frac{1}{3}:\frac{1}{2}:1$; $\frac{1}{5}:\frac{1}{4}:\frac{1}{3}:\frac{1}{2}$; $a:b:c:d$.

7. From a point in a plane four lines (called rays) are so drawn as to give four angles just covering the plane around the point. The angles are in the ratio $1:2:3:4$. Find them.

8. Find the four angles described in Problem 7 if their ratios are the following: (1) $2:3:5:6$; (2) $3:7:9:11$; (3) $\frac{1}{2}:\frac{2}{3}:\frac{3}{4}:\frac{4}{5}$; (4) $a:b:c:d$; (5) $1:a:4:3a$.

9. Three angles just cover all of the plane around a point. The difference of the first and second is 30°, and of the second and third is 30°. Find them.

10. Three angles just covering the plane around a point are to each other as $2:3:4$. Find them.

11. The sum of the first and second of three angles that just cover the plane around a point is 200° and the difference of the third and first is 80°. Find them.

12. Two straight lines intersect at a point, $O$, making four angles, one of which is $a°$ and another $3a°$. Find the value of $a$.

13. Two parallels are cut by a transversal, giving angles as shown here. Find $x$ and $y$ and all eight of the angles.

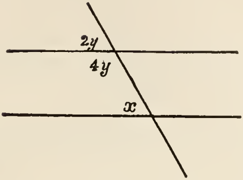

FIG. 1

14. Two parallels are cut by a transversal making angles as shown here; find $x$ and $y$, and all eight of the angles.

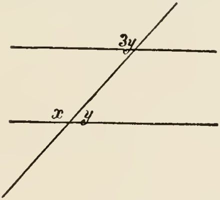

FIG. 2

15. Two parallels are cut by a transversal making two adjacent angles, $x$ and $y$, having the ratio of 3 : 4. Find the eight angles in degrees.

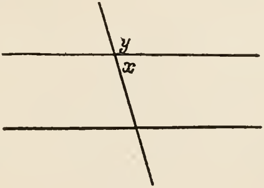

FIG. 3

16. Two straight lines are cut by a transversal making the two interior angles, $x$ and $y$, Fig. 4, on the same side of the transversal $87\frac{1}{2}°$ and $93\frac{1}{2}°$ respectively. Are the lines parallel?

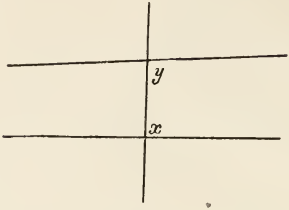

Fig. 4

17. Two parallel lines, cut by a transversal, give interior angles, $x$ and $y$, on the same side of the transversal, whose difference is 25°. Find $x$ and $y$ and the remaining six angles.

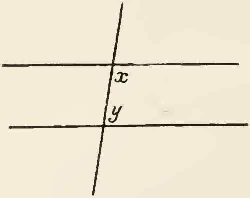

Fig. 5

18. In the figure below find the values of $x$ and $y$.

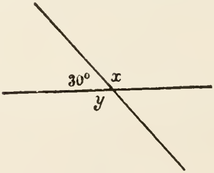

Fig. 6

19. A pair of lines cut by a transversal gives angles of the magnitudes shown in the following figure. Are the lines parallel?

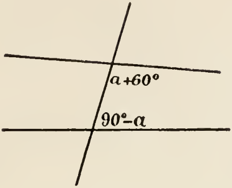

FIG. 7

20. Different pairs of lines are cut by a transversal giving in succession the following angular relations. Tell, in each case, whether the lines are parallel and why.

(1) Interior angles on same side of transversal, $45°$ and $135°$;

(2) Interior angles on same side of transversal, $x+160°$, and $20°-x$;

(3) Interior angles on same side of transversal, $x+a-10°$ and $a-x+190°$;

(4) Alternate interior angles, $75°$ and $74^{\circ}.9$;
(5) " " " $a-60°$ and $240°-a$;
(6) " " " $x+y$ and $y+x$;
(7) " " " $x-y$ and $y-x$;
(8) " " " $90°-c$ and $90°+c$;
(9) " " " $-a+75°$ and $75°-a$.

## §6. Of Angles of a Scalene Triangle

1. The angles of a triangle are $x$, $y$, and $z$, and $x+y=80°$ and $z-y=45°$. Find $x$, $y$, and $z$.

2. The angles of a triangle are $x$, $2x$, and $y$, and the angle, $y$, is $8°$ larger than $x$. Find all the angles in degrees.

3. The angles of a triangle are $x$, $y$, and $3y$, and $x$ is $35°$ larger than $y$. Find all the angles in degrees.

4. The angles of a triangle are $5x$, $20x$, and $20°$. Find the unknown angles.

5. The angles of a triangle are $2x$, $y$, and $z$. The exterior

angle of the triangle that is adjacent to $2x$ is 94°, and the difference of the other two angles of the triangle is 26°. Find all the angles of the triangle.

6. The angles of a triangle are as 1 : 2 : 3. Find them.

7. Find the angles of a triangle if their ratios are 3 : 5 : 7; 4 : 5 : 9; 6 : 7 : 11; $\frac{1}{4} : \frac{1}{3} : \frac{1}{2}$; $a : b : c$; $a : 3a : 5a$.

8. Two angles of a triangle are $3x$ and $2y$ and the exterior adjacent angles are 66° and 100°, respectively. Find all the angles of the triangle.

9. The angles of a triangle are $x$, $2x+10°$, and $3x-15°$. Find them.

10. The angles of a triangle may be denoted by $3x^2$, $9x$, and $3x$. Find $x$ and each of the angles of the triangle.

11. Find, in degrees, $x$ and each of the angles of the triangles, whose separate angles are denoted by the following triplets of numbers:

(1) $3x^2$, $10x$, $2x$;
(2) $3x^2$, $6x$, $6x$;
(3) $3x^2$, $8x$, $4x$;
(4) $15x^2$, $40x$, $20x$;
(5) $15x^2$, $30x$, $30x$;
(6) $15x^2$, $45x$, $15x$;
(7) $4x^2$, $15x$, $x$;
(8) $4x^2$, $12x$, $4x$;
(9) $4x^2$, $10x$, $6x$;
(10) $4x^2$, $8x$, $8x$;
(11) $9x^2$, $50x$, $4x$;
(12) $9x^2$, $44x$, $10x$;
(13) $9x^2$, $27x$, $27x$;
(14) $9x^2$, $30x$, $24x$;
(15) $20x^2$, $100x$, $20x$;
(16) $20x^2$, $80x$, $40x$;
(17) $20x^2$, $60x$, $60x$;
(18) $36x^2$, $100x$, $44x$;
(19) $36x^2$, $72x$, $72x$;
(20) $60x^2$, $100x$, $20x$;
(21) $60x^2$, $90x$, $30x$;
(22) $60x^2$, $60x$, $60x$.

12. The angles of a triangle are $x+y$, $x-y$, and 16°. Find all of them.

## §7. Of Angles of a Right Triangle

1. The acute angles of a right triangle are $x$ and $y$. Show the relation between $x$ and $y$ by an equation, and solve the equation (1) for $x$; (2) for $y$.

2. The acute angles of a right triangle are $x$ and $3x$. Find them.

3. The acute angles of a right triangle are $3x$ and $6x$. Find them.

4. The acute angles of a right triangle are $2x$ and $\frac{x}{2}$. Find them.

5. The acute angles of a right triangle are $\frac{y}{2}$ and $\frac{y}{3}$. Find $y$ and the acute angles.

6. The acute angles of a right triangle are $ax$ and $bx$. Find them.

7. The acute angles of a right triangle are $\frac{z}{c}$ and $\frac{z}{d}$. Find them.

8. An exterior angle to one of the acute angles of a right triangle is 140°. Find the acute angles of the triangle.

9. The acute angles of a right triangle are $3x+18°$ and $6x-48°$. Find the acute angles.

10. The difference between the two acute angles of a right triangle is 26°. Find the angles.

11. The acute angles of a right triangle are $3z+a$ and $6z-b$. Find the acute angles.

12. The difference between the acute angles of a right triangle is 25°. Find them.

13. The difference between the two acute angles of a right triangle is $a°$. Find them.

14. $\frac{x}{a}$ and $\frac{y}{b}$ denote the acute angles of a right triangle whose difference is 24°. Find the angle $x$ and the acute angles.

15. $(a+b)x$ and $(a-b)x$ are the acute angles of a right triangle. Find them.

16. $(a+b)x$ and $(a+b)y$ denote two acute angles of a right triangle, whose difference is $a^2-b^2$. Find the acute angles.

17. $9x$ and $7y$ are two acute angles of a right triangle, and $x-y=2\frac{8}{9}°$. Find the angles.

18. $\frac{y}{a-b}$ and $\frac{z}{a+b}$ are the acute angles of a right triangle and their difference is $c°$. Find $y$ and $z$ and the angles of the triangle.

19. The acute angles of a right triangle are as $2:5$. Find them.

20. The acute angles of a right triangle are as $\frac{3}{4}:\frac{5}{6}$. Find them.

21. The acute angles of a right triangle are as $a:b$. Find them.

22. One acute angle of a right triangle is $n$ times the other. Find the acute angles.

23. The acute angles of a right triangle are equal. Find them.

24. The exterior angle to an acute angle of a right triangle is $d°$. Find the acute angles of the triangle.

25. The angles of a triangle are $x$, $y$, and $z$. A perpendicular from the vertex of $z$ upon a bisector of $x$ makes with the sides $ZX$, and $ZY$, the angles $\frac{1}{2}(x+y)$ and $x-y$, respectively. Find in degrees the angles $x$, $y$, and $z$.

26. Solve Problem 25 if instead of $\frac{1}{2}(x+y)$ and $x-y$, the angles had been $\frac{3}{4}(x+y)$ and $2(x-y)$, respectively.

27. By the principle of vertical angles and of the acute angles of a right triangle, find $x$, $y$, and the acute angles of the right triangle of Fig. 8, if the angles may be designated as shown.

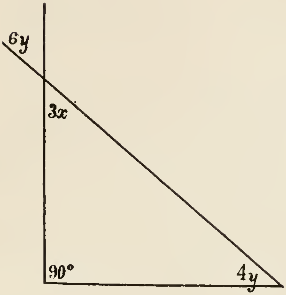

Fig. 8

28. By the principles suggested in Problem 27, find $x$, $y$ and the acute angles of the right triangle of Fig. 9.

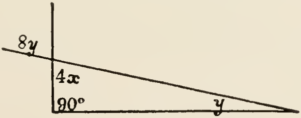

Fig. 9

29. Similarly, find $x$, $y$, and the acute angles of the right triangle of Fig. 10.

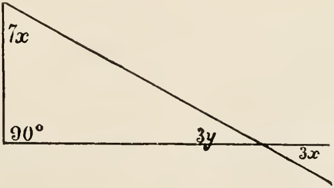

Fig. 10

30. In like manner find $x$, $y$, and the acute angles of the right triangle of Fig. 11.

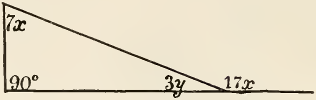

Fig. 11

31. The acute angles of a right triangle may be denoted by the following algebraic expressions: find $x$, and the acute angles of each triangle:

(1) $3x$ and $6(x-15^\circ)$;
(2) $15x$ and $6(x+1^\circ)$;
(3) $18x-30^\circ$ and $2(2x+5^\circ)$;
(4) $9x+23^\circ$ and $5(5x-3^\circ)$;
(5) $22x+7^\circ$ and $6(6x+1^\circ)$;
(6) $\frac{2x}{5}$ and $x+6^\circ$;
(7) $9x-8^\circ$ and $\frac{7x+46^\circ}{2}$;
(8) $\frac{2x+34^\circ}{11}$ and $4x+\frac{320^\circ}{x}$;
(9) $\frac{4x}{7}+38^\circ$ and $\frac{5x}{3}+\frac{525^\circ}{x}$;
(10) $5\left(\frac{x}{6}+1^\circ\right)$ and $7\left(\frac{x}{9}+8^\circ\right)$.

## §8. Of Angles of an Isosceles Triangle

1. The angles of an isosceles triangle are $x$, $x$, and 68°. Find the base angles.

2. The angles of an isosceles triangle are $y$, $y$, and $4y$. Find them.

3. The angles of an isosceles triangle are $z$, $z$, and $az$. Find them.

4. The angles of an isosceles triangle are $b$, $b$, and $b$. Find them all in degrees.

5. The vertex angle of an isosceles triangle is 60°. Find the other two in degrees.

6. One base angle of an isosceles triangle is 60°. Find the other two angles.

7. The vertex angle of an isosceles triangle lacks 5° of being $\frac{2}{9}$ of a right angle. Find all the angles of the triangle.

8. Find the angles of an isosceles triangle when one of the angles at the base is equal to $\frac{1}{4}$ the vertex angle.

9. Find the angles of an isosceles triangle in which the vertex angle is $\frac{2}{5}$ of the sum of the base angles.

10. Find the angles of an isosceles triangle in which the vertex angle is 12° less than $\frac{1}{5}$ the sum of the base angles.

11. The vertex angle of an isosceles triangle is $\frac{7}{8}$ of an angle at the base. Find all the angles of the triangle.

12. The vertex angle of an isosceles triangle is 42° greater than $\frac{5}{8}$ of an angle at the base.

13. The angle at the vertex of an isosceles triangle is $\frac{1}{3}$ the exterior angle at the vertex. How many degrees in each angle, exterior and interior, at the base?

14. Find all the angles of an isosceles triangle (exterior and interior):

(1) If the vertex angle is $50°$; $80°$; $67°$; $62°30'$; $m°$; $d°$;

(2) If the angle exterior to the vertex angle is $130°$; $129°$; $135°$; $90°+a°$; $d°$;

(3) If a base angle is $45°$; $64°$; $39°$; $22°30'$; $180°-a°$; $d°$;

(4) If the angle exterior to a base angle is $100°$; $120°$; $163°$; $180-b°$; $d°$.

15. The exterior angles of an isosceles triangle are $x$, $x$, and $y$ degrees and the ratio of $x:y$ is $2:3$. Find the angles of the triangle.

16. The vertex angle of an isosceles triangle lacks 8° of being $\frac{3}{5}$ of $\frac{7}{9}$ of a right angle. Find it.

17. The vertex angle of an isosceles triangle is $n$ times a base angle. Find all the angles in degrees.

18. A base angle of an isosceles triangle is $n$ times the vertex angle. Find all the angles.

19. The vertex angle of an isosceles triangle is $\frac{1}{n}$ of one of the base angles. Find all the angles of the triangle.

20. One base angle of an isosceles triangle is $\frac{1}{n}$ of the vertex angle. Find all the angles.

21. The vertex angle of an isosceles triangle is $\frac{1}{n}$ of the sum of the two base angles. Find all the angles of the triangle.

22. The base angles of an isosceles triangle may be denoted by each of the following pairs of algebraic expressions, $x$ denoting the same number in both expressions of each pair. Find the value in degrees of all angles of the several triangles:

(1) $12x+7^\circ$ and $9x+13^\circ$;
(2) $17x-50^\circ$ and $2x-5^\circ$;
(3) $27x+4^\circ$ and $11x+28^\circ$;
(4) $21^\circ-7x$ and $2x+57^\circ$;
(5) $29x-11^\circ$ and $19^\circ-11x$;
(6) $8x+19^\circ$ and $25x-32^\circ$;
(7) $5x+14^\circ$ and $17^\circ-2x$;
(8) $19x$ and $84^\circ-9x$;
(9) $78^\circ-x$ and $8x-69^\circ$;
(10) $\frac{5x-25^\circ}{3}$ and $50^\circ-\frac{5x}{4}$;
(11) $\frac{4x+7^\circ}{3}$ and $2x+25^\circ$;
(12) $\frac{2x}{5}-\frac{x}{3}$ and $10^\circ-\frac{x-3}{6}$;

(13) $\frac{x}{5}+\frac{x}{2}$ and $\frac{3x}{10}-\frac{x}{4}+6\frac{1}{2}$;

(14) $\frac{x}{3}+\frac{x}{6}$ and $\frac{x}{4}+18°$;

(15) $\frac{7x}{2}-\frac{3x}{4}$ and $\frac{3x}{5}+8\frac{3}{5}°$;

(16) $x-\frac{x}{7}+20°$ and $\frac{3x}{4}+26°$.

23. Find each of the angles of an isosceles triangle, whose several angles may be represented by the following sets of three algebraic expressions, $x$ denoting the same number throughout each set:

(1) $x+18°$, $x+18°$, and $10x+24°$;

(2) $\frac{3x-9°}{5}$, $\frac{3x-9°}{5}$, and $3x+3°$;

(3) $40°-x$, $40°-x$, and $3(20°-2x)$;

(4) $\frac{7x+15°}{2}$, $\frac{7x+15°}{2}$, and $3(2x-10°)$;

(5) $\frac{2x}{3}-\frac{60}{x}$, $\frac{2x}{3}-\frac{60}{x}$, and $\frac{5(x-18°)}{2}-\frac{180°}{x}$;

(6) $\frac{3x+55°}{4}$, $\frac{3x+55°}{4}$, and $\frac{4x}{15}$;

(7) $\frac{10x}{7}$, $\frac{10x}{7}$, and $\frac{9x-52°}{2}$;

(8) $9\left(x-\frac{20}{x}\right)$, $9\left(x-\frac{20}{x}\right)$, and $4(x-1)$;

(9) $\frac{7x-26°}{4}$, $\frac{7x-26°}{4}$, and $7x+\frac{72°}{x}$;

(10) $\frac{5}{2}x-\frac{108°}{x}$, $\frac{5}{2}x-\frac{108°}{x}$, and $\frac{216°}{x}$.

24. $x$ and $y$ stand for the same numbers in both triangles of Fig. 12. Find $x$, $y$, and all the unknown angles.

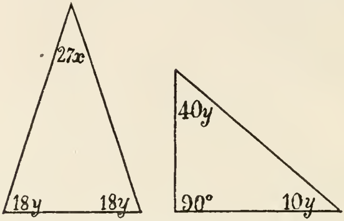

Fig. 12

25. By the principle of opposite angles, and of the angles of an isosceles triangle, find $x$, $y$, and the angles of an isosceles triangle, if they may be designated as shown in Fig. 13.

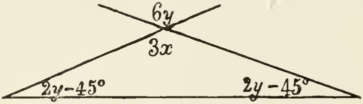

Fig. 13

26. Find the angles of the isosceles triangle of Fig. 13, by the principles suggested in Problem 25.

27. Similarly, find the angles of the isosceles triangle of Fig. 14.

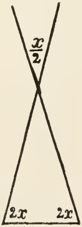

Fig. 14

28. In like manner, find the angles of the isosceles triangle of Fig. 15.

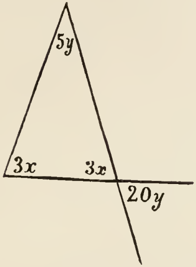

FIG. 15

## §9. Of Angles of Polygons

1. How many sides has a polygon, the sum of whose angles is 7 R. A. (R. A.=right angle)? 9 R. A.? 10 R. A.? 18 R. A.? 19 R. A.? 24 R. A.? 30 R. A.?

2. How many sides has a polygon, the sum of whose angles is 40 R. A.? $n$ R. A.? $n+2$ R. A.? $r$ R. A.? $r+3$ R. A.? $n+r$ R. A.? $n+r+2$ R. A.?

3. An interior angle of a regular polygon is 170°. How many sides has the polygon?

4. Each interior angle of a regular octagon is $a$. Find $a$ in degrees.

5. Each interior angle of a regular dodecagon is $x$. Find $x$ in degrees.

6. Each interior angle of a regular 18-gon is $y$. Find $y$ in degrees.

7. Each interior angle of a regular $n$-gon is $z$. Find $z$ in degrees.

8. Each exterior angle of a regular hexagon is $a$. Find $a$ in degrees.

9. Each interior angle of a regular icosagon (20-sided polygon) is $p$. Find $p$ in degrees.

10. Each exterior angle of a regular $n$-gon is $x$. Find $x$ in degrees.

11. The exterior angle of a regular polygon is 10°. How many sides has the polygon?

12. An interior angle of a regular polygon is 160°. How many sides has the polygon?

13. An interior angle of a regular polygon is 165°. How many sides has the polygon?

14. An interior angle of a regular polygon is $\frac{n-2}{n}\times 180°$. How many sides has the polygon?

15. How many sides has a polygon, the sum of whose interior and exterior angles is 1260°? 2880°? 3960°? 5040°? $36\times 2$ R. A.? $2n$ R. A.? $(2n-4)$R. A.? $(2n+4)$ R. A.?

16. How many sides has a regular polygon the sum of whose interior angles minus the sum of its exterior angles is 180°? 540°? 1080°? 2880°? 3600°? $(2n-8)$ R. A.?

17. How many sides has any polygon the sum of whose interior angles minus the sum of its exterior angles is 1980°? 2520°? 4680°? 8280°? $(2n-8)\times 90°$?

18. Prove, algebraically, that if two angles, $x$ and $y$, of a quadrilateral are supplementary, the other two, $a$ and $b$, are also supplementary.

19. Find all the integral values which the exterior angle of a regular polygon can have. (Use $\frac{360°}{n}$ = an integer.)

20. Can the following angular values be the exterior angles of a regular polygon: 56°? 72°? 85°? 38°? 62°? 144°? 45°? 36°? 89°? $\frac{360}{n}$?

21. An exterior angle of a regular polygon equals $\frac{2}{15}$ of the adjacent angle. How many sides has the polygon?

22. How many sides has a regular polygon if an exterior angle equals the following fractional parts of the adjacent interior angle: $\frac{2}{3}$? $\frac{1}{4}$? $\frac{1}{12}$? $\frac{2}{35}$? $\frac{1}{24}$? $\frac{2}{63}$? $\frac{1}{40}$? $\frac{2}{99}$? $\frac{1}{60}$? $\frac{2}{m^2-1}$ (where $m$ is any integer)?

23. The interior angle of a regular polygon equals $\frac{m^2-1}{2}$ ($m$ any integer) of an exterior angle. How many sides has the polygon?

24. The numerical measure of one angle of a parallelogram equals $\frac{1}{2}$ of the square of the numerical measure of the adjacent angle. Find each angle.

## § 10. Of Angles Measured by Arcs of Circles

1. If the angle at $O$, Fig 16, contains $8x-18$ degrees, how many degrees are there in the angle at $C$?

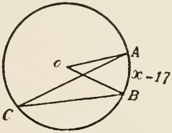

FIG. 16

2. If there are $3x-\frac{7}{2}$ degrees in the angle at $C$, Fig. 16, how many degrees are there in the angle at $O$?

3. If the arc, $AB$, Fig. 16, which is $x-17$ degrees, measures $\frac{1}{7}$ of the circumference, how many degrees are there in the arc, $ACB$?

4. If there are $7a$ degrees in the arc $AC$, Fig. 17, how many

degrees of angle are there in the angle $TAC$, between the tangent $AT$ and the chord $AC$?

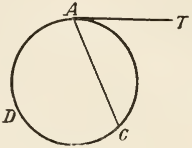

FIG. 17

5. If there are $16y-130$ degrees in the arc $AC$, Fig. 17, how many degrees are there in the arc $ADC$?

6. If two chords intersect as shown in Fig. 18, and the arcs, in degrees, are as indicated by the letters beside them and the angle 80° lies as shown, find $x$ and $y$.

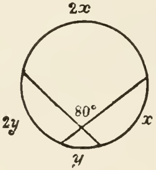

FIG. 18

7. If the letters about the circumference, Fig. 19, indicate the number of degrees in the arcs as shown, and the secants inclose an angle of 48°, find $x$ and $y$.

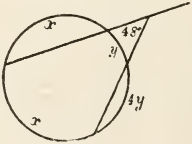

FIG. 19

8. With angle and arcs as shown in Fig. 20, find $x$.

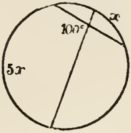

FIG. 20

9. With angle and arcs as shown in Fig. 21, find the number of degrees in the angle between the secants.

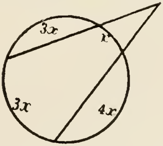

FIG. 21

10. The angle between the two tangents of Fig. 22, is 60°; find the arcs $x$ and $y$.

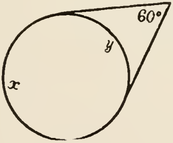

FIG. 22

11. The outside angle between the two tangents is 128° (see Fig. 23). Find the number of degrees in $x$ and $y$.

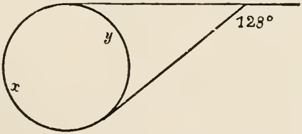

FIG. 23

12. Find the arcs $x$ and $y$, intercepted by two intersecting tangents which make (1) an inside angle of 40°; of 80°; of 120°; of $a$°; (2) an outside angle of 150°; of 210°; of 50°, of $c$°.

13. Find the arc, $x$, in degrees, that an intersecting tangent and secant make, if the arcs and the angle between the tangent and secant have the values indicated below (see Fig. 24 for the first):

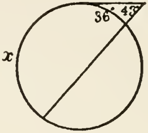

FIG. 24

| Farther Arc | Nearer Arc | Angle |
|---|---|---|
| $x$ | $36°$ | $43°$; |
| $2x$ | $20°$ | $88°$; |
| $10x-36°$ | $16°$ | $44°$; |
| $\frac{8x-9°}{5}$ | $15°$ | $55\frac{3}{5}°$; |
| $x$ | $2a°$ | $a°$; |
| $x$ | $b°$ | $2a°$. |

14. Find the angle between an intersecting tangent and secant, if the lengths of the arcs are as shown in Fig. 25.

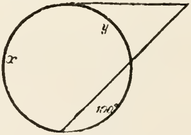

FIG. 25

15. Find the angle between an intersecting tangent and

secant (Fig. 26 for the first), if the intercepted arcs and the outside arc have the following values:

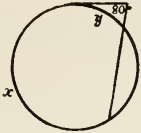

FIG. 26

| Farther Arc | Nearer Arc | Outside Arc |
|---|---|---|
| $x$ | $y$ | 80°; |
| $x$ | $y$ | 90°; |
| $x$ | $y$ | 160°; |
| $x$ | $y$ | $a$°. |

16. Two tangents make a right angle with each other. Into what arcs do their points of tangency divide the circumference? Denote the arcs by $x$ and $y$.

17. A semi-circumference, *ABC*, is divided at *C* into arcs having a ratio to each other of 3 : 4. Find the angle formed by the diameter *AB*, and the chord, *CB*.

18. A tangent and secant passing through the center of the circle make an angle of 47°. Find the two arcs intercepted. Designate the arcs by $x$ and $y$.

19. A circumference is divided at *A*, *B*, *C*, and *D* into four arcs in the ratio of 1 : 2 : 3 : 4. Find the angles formed by the chords *AC* and *BD*.

20. Find the number of degrees in the angle formed by two tangents intercepting an arc of 205°.

21. Two tangents from a point intercept on a circle arcs that are in the ratio of 1 : 3. Find the angle between the tangents in degrees.

22. The angle between two secants intersecting without the circumference, is 76°. One of the intercepted arcs is 245°. Find the other.

23. Find the value of a central angle standing on an arc that is equal to the radius.

24. Find the angle subtended at the center of a circle by an arc of 15 feet, the radius being 20 feet.

## §11. Of Chords Intersecting within Circles

1. Two chords intersect within a circle at a point which divides one into sects $x$ inches, and 3 inches and the other into the sects 2 inches and 12 inches. Find $x$.

2. The sects of one of two intersecting chords are 4 and $x$, and of the other, 2 and 10. Find the unknown sect, $x$.

3. The sects of one of two intersecting chords are $a$ and $x$, and of the other 4 and $a$. Find the sect $x$.

4. The sects of one of two intersecting chords are 4 and 4, and the entire length of the other chord is 10. Find the sects of the second chord. Interpret both results.

5. The length of the first of two intersecting chords is 9 and the sects of the other are 2 and 9. Find the sects of the first chord. Show by sketch the meaning of both results.

6. The length of one of two intersecting chords is 13 and the sects of the other are 4 and $5\frac{1}{2}$. Find the sects of the first chord. Interpret all results.

7. The sects of one of two intersecting chords are $y$ and $13-y$, and of the other, 4 and 10. Find $y$, and the length of the first chord. Interpret all results.

8. The sects of one of two intersecting chords are $z$ and $l-z$, and of the other, 5 and 8. Find $z$ and the length of the first chord. Interpret results.

9. The sects of one of two intersecting chords are $z$ and $l-z$, and of the other, 6 and $\frac{1}{6}$. Find the sects of the first chord in terms of $l$. Interpret results.

10. The sects of one of two intersecting chords are $z$ and $a-z$, and of the other, $n$ and $\frac{a}{n}$. Find the values of $z$ in terms of $a$. Interpret results.

11. The sects of one of two intersecting chords are $x+1$ and $x+5$, and of the other, 3 and 4. Find $x$ and the length of the first chord. Interpret results.

12. Two chords intersect within a circle. Find $x$ and the length of the first chord under the conditions indicated below and interpret results:

Sects of:

| | First Chord | Second Chord |
|---|---|---|
| (1) | $x-1$ and $x+9$, | 3 and 8; |
| (2) | $x-1$ and $x+4$, | 2 and 9; |
| (3) | $x-5$ and $x+3$, | 4 and 5; |
| (4) | $x-8$ and $x-7$, | 2 and 15; |
| (5) | $x+2$ and $x+5$, | $4\frac{2}{3}$ and 6; |
| (6) | $x-3$ and $x-12$, | 4 and $17\frac{1}{2}$; |
| (7) | $2x-1$ and $6x+1$, | 6 and $6\frac{1}{2}$; |
| (8) | $2x+2$ and $3x-2$, | 4 and 14; |
| (9) | $6x+4$ and $8x-3$, | 4 and $12\frac{1}{2}$; |
| (10) | $ax+b$ and $ax-b$, | $a+b$ and $a-b$; |
| (11) | $x+a$ and $3x-2a$, | $a$ and $2a$; |
| (12) | $3x-c$ and $5x-3c$, | $c$ and $4c$. |

13. The sects of the first of two intersecting chords are $x+y$ and $x-y$ and of the second 6 and 9. The sect $x+y$ is 18. Find $x$ and $y$ and the length of the first chord. Interpret both results.

14. The sects of the first of two intersecting chords are $x+y$ and $x-y$ and of the second 5 and 12. The sect $x-y$ is 6. Find $x$ and $y$ and the length of the first chord.

15. The sects of intersecting chords and the value of one of these indicated as a sum or difference are given below. In

each case, find $x$ and $y$ and the length of the first chord and interpret results.

Sects of:

| | First Chord | Second Chord and the Given Sect | |
|---|---|---|---|
| (1) | $x+y$ and $x-y$, | 2 and 26, | $x+y=13$; |
| (2) | $x+y$ and $x-y$, | 3 and 21, | $x-y=7$; |
| (3) | $x+y$ and $x-y$, | 8 and 8, | $x+y=16$; |
| (4) | $x+y$ and $x-y$, | 2 and 32, | $x-y=4$; |
| (5) | $x+y$ and $x-y$, | 5 and 13, | $x-y=4$; |
| (6) | $x+y$ and $x-y$, | 6 and 11, | $x-y=3$. |

16. The sects of the first of two intersecting chords are $x$ and $y$ and of the second 3 and 4. The length of the first chord is 8. Find the sects $x$ and $y$.

17. The sects and the length of the first of two intersecting chords and the sects of the first chord are indicated below. In each case find the unknown sects:

| | Sects of First Chord | Length of First Chord | Sects of Second Chord |
|---|---|---|---|
| (1) | $x$ and $y$, | 18, | 4 and 8; |
| (2) | $x$ and $y$, | 15, | 4 and 9; |
| (3) | $x$ and $y$, | 18, | 5 and 9; |
| (4) | $x$ and $y$, | 28, | 5 and 15; |
| (5) | $x$ and $y$, | 30, | 9 and 9; |
| (6) | $x$ and $y$, | 26, | 8 and 11; |
| (7) | $x$ and $y$, | 19, | 4 and 21; |
| (8) | $x$ and $y$, | 19, | 2 and 45. |

18. The sects of the first of two intersecting chords are $x+4$ and $x-5$ and of the second $x+1$ and $x-4$. Find the value of $x$, and of the four sects.

19. The sects of two intersecting chords are as given below. Find $x$ and the length of each of the two chords.

Sects of:

| | First Chord | Second Chord |
|---|---|---|
| (1) | $x-3$ and $x+7$, | $x-2$ and $x+3$; |
| (2) | $x+1$ and $x+5$, | $x-3$ and $x+17$; |

(3) $x-9$ and $x-1$, $x-6$ and $\frac{2x-13}{2}$;

(4) $2x$ and $\frac{3x+8}{4}$, $3x-11$ and $x+10$;

(5) $4x-\frac{3}{2}$ and $3x-\frac{7}{2}$, $2x$ and $8x-3$;

(6) $\frac{3x}{5}+6$ and $2(x+10)$, $x-10$ and $2x$.

20. The sects of one of two intersecting chords are 2 and $x+2y$ and of the other, $3\frac{1}{2}$ and 4. The sects of the first chord made by a third chord are 1 and $x-2y$ and of the third chord, 2 and 4. Find $x$ and $y$.

21. The sects of one of two intersecting chords, $AB$ and $CD$, Fig. 27, are $x+y$ and $x-y$ and of the other 5 and 9. Chord $AB$ stands stationary and the circle moves so that $AK$ and $EF$ are intersecting chords, the sects of $AK$ being $x$ and $y$ and of $EF$, 2 and 15. Find $x$ and $y$.

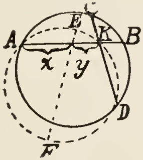

Fig. 27

22. Find $x$ and $y$ if the relations of chords and circles like those of Fig. 27 gave sects $x+y$ and $x-y$ with 8 and 11, and $x$ and $y$ with 2 and $16\frac{1}{2}$. ($KB=x-y$.)

### §12. Of Tangents and Secants of Circles

1. A tangent and a secant to the same circle intersect. The length of the tangent is $x$, of the whole secant $x+5$, and of the outside sect $x-2$. Find the lengths of all the lines. Express surds as radicals in simplest form. Also find, to two decimal places, their values.

2. Solve exercises like the latter for the following sets of data.

Length of:

| | Tangent | Whole Secant | Outside Segment |
|---|---|---|---|
| (1) | $x$ | $x+10$ | $x-3$; |
| (2) | $x$ | $x+5$ | $x-5$; |
| (3) | $x$ | $x+2$ | $x-4$; |
| (4) | $x$ | $x+12$ | $x+4$; |
| (5) | $x$ | $x-4$ | $x-12$; |
| (6) | $x$ | $\frac{7x+12}{7}$ | $x-4$; |
| (7) | $x$ | $x+5$ | $\frac{3x+10}{3}$; |
| (8) | $x$ | $\frac{8x+15}{8}$ | $x-5$; |
| (9) | $x$ | $x+12$ | $x-4$. |

3. A tangent intersects a secant through the center of a circle. The length of the tangent is 18, and that of the outside sect of the secant is 6. Find the radius of the circle and the length of the secant.

4. A tangent intersects a secant through the center of a circle. The lengths of the tangent and of the outside sect of the secant are given below. Find the radius, $r$, of the circle and the whole secant in each case:

| | Tangent | Outside Sect | | Tangent | Outside Sect |
|---|---|---|---|---|---|
| (1) | 4 | 2; | (5) | 9 | 1; |
| (2) | 5 | 3; | (6) | 13 | 3; |
| (3) | 7 | 3; | (7) | 15 | 5; |
| (4) | 9 | 7; | (8) | $t$ | $s$. |

5. With a tangent and secant as in Problem 4, and with data as indicated below, find the length of the tangent, in each case. When results are irrational, express them (1) as radicals in their simplest form, and (2) arithmetically, to two decimal figures. What is the significance of the double result?

| | Radius | Outside Sect | | Radius | Outside Sect |
|---|---|---|---|---|---|
| (1) | 10 | 4; | (6) | 30 | 5; |
| (2) | 4 | 4; | (7) | $r$ | $s$; |
| (3) | 16 | 20; | (8) | $\frac{d}{2}$ | $\frac{s}{10}$; |
| (4) | 48 | 12; | | | |
| (5) | 12 | 3; | (9) | $m$ | $n$. |

6. Given a tangent and secant as in Problem 4, express the outside sect, $s$, of the secant in terms of the length, $l$, of the tangent, and of the radius, $r$, of the circle.

7. Given $l=9$ and $r=12$, $l$ and $r$ defined as in Problem 6; find $s$. Explain the meaning, if there is any, of the double result.

8. If two secants intersect outside of a circle, prove algebraically that the whole secants, $a+b$ and $c+d$, are reciprocally proportional to their outside sects, $a$ and $c$.

9. If two secants intersect outside of a circle and give outside and inside sects as indicated below, find the unknown sects, and state what double values signify:

| | SECTS OF FIRST SECANT | | SECTS OF SECOND SECANT | |
|---|---|---|---|---|
| | Outside | Inside | Outside | Inside |
| (1)... | $x$ | 8 | 5 | 16 |
| (2)... | $x$ | 24 | 5 | 40 |
| (3)... | $x$ | 42 | 9 | 6 |
| (4)... | $x$ | 26 | 16 | 14 |
| (5)... | 10 | $x$ | 15 | 5 |
| (6)... | 14 | $x$ | 12 | 58 |
| (7)... | 15 | $x$ | 24 | 6 |
| (8)... | $a$ | $x$ | $c$ | $d$ |
| (9)... | 9 | 11 | $x$ | 8 |
| (10)... | 5 | 3 | $x$ | 1 |
| (11)... | 7 | 23 | $x$ | 11 |
| (12)... | 9 | 11 | $x$ | 24 |
| (13)... | 19 | 11 | 10 | $x$ |
| (14)... | 13 | 17 | 10 | $x$ |
| (15)... | 20 | $b$ | 10 | $x$ |
| (16)... | $a$ | $b$ | $c$ | $x$ |

10. From a point outside of a circle a tangent and a secant are drawn to the circle. The outside sect of the secant is 7 inches shorter and the chord is 3 inches shorter than the tangent. Find the length of the tangent.

## §13. Of Properties of Squares and Rectangles

### SQUARES

1. Denoting the side of a square by 10 and the diagonal by $x$, express $x$ (1) as a radical in its simplest form; and (2) as a decimal to two significant decimal figures.

2. Denoting the diagonal of a square by $x$ and the side by $a$, express $x$ in terms of $a$.

3. Denoting the side of a square by $x$ and the diagonal by 20, express $x$ (1) as a radical in its simplest form; and (2) as a decimal to two places.

4. Denoting the side of a square by $x$ and the diagonal by $d$, express $x$ in terms of $d$.

5. Denote the side of a square by $a$, the diagonal by $d$, and the area by $A$. Supposing any one of these is given find the other two.

6. The side of a square is 15. Calculate the sides and the area of an isosceles triangle, made by connecting the middle of one side of the square with the ends of the opposite side.

7. Solve Problem 6, supposing the side of the square to be $a$.

8. Calculate the sides and the area of an isosceles triangle made as described in Problem 6, supposing the length of the diagonal to be 16; 24; 45; 49; 100; $a$; $a+b$.

9. Calculate the sides of the same isosceles triangle, supposing the area of the square to be 1600; 3600; 8100; $2A$; $4(a-b)^2$.

10. Calculate the radius of the circle inscribed in, and of the circle circumscribed about a square whose area is 1600; 5000; $a^2$; $A$.

11. The side of one square is 3 times as long as that of another square, and its area is 72 square yards greater than that of the second square. How long is the side of the first square?

12. What is the area of a square whose diagonal is 1 inch longer than the side?

13. How long is the side of a square whose area increases by 25 square yards while the side increases by 1 yard?

14. The length of the side of one square is 9 yards less than 8 times the length of another, and the area of the former is 90 square yards greater than that of the latter. Find the area of both squares.

### RECTANGLES

1. Denoting the altitude of a rectangle by $a$ and the base by $3a$, express the area, $A$, and the perimeter, $p$.

2. Express the area, $A$, and the perimeter, $p$, of a rectangle of altitude $a+8$ and of base $a+18$; of altitude $2x-3$ and base $2x+3$; of altitude $3x-5$ and base $5x+3$; of altitude $x-y$ and base $x+y$; of altitude $3x-2y$ and base $2x+3y$; of altitude $12x-7$ and base $12y+7$; of altitude $a(x+3)$ and base $a(x+3)$.

3. Calculate the diagonal, $d$, of a rectangle whose dimensions are 21 and 28; 18 and 36; 12 and 40.

4. Express the diagonal, $d$, of a rectangle whose dimensions are $a$ and $b$; $3a$ and $4a$; $3a+3$ and $4a+4$; $a-b$ and $a+b$; $2x-3y$ and $2x+3y$; $a-b$ and $a-b$; $a+b$ and $a+b$.

5. Given the following pairs of values of the area, $A$, and the diagonal, $d$, of a rectangle, find the sides $x$ and $y$:

$$(1)\begin{cases} A=18; \\ d=8; \end{cases} \qquad (3)\begin{cases} A=8a^2; \\ d=5a; \end{cases} \qquad (5)\begin{cases} A=ab; \\ d=a-b; \end{cases}$$

$$(2)\begin{cases} A=18; \\ d=10; \end{cases} \qquad (4)\begin{cases} A=ab; \\ d=a+b; \end{cases} \qquad (6)\begin{cases} A=a; \\ d=c. \end{cases}$$

6. Denoting the sides of a rectangle by $a$ and $b$, the diagonal by $d$, and the area by $A$, and having given any two of these numbers, find the other three.

7. The area of a rectangle is 400 and half its perimeter is 41. Find the base, $b$, and the altitude, $a$.

8. If a rectangle is 7 feet longer than it is wide, and it contains 170 square feet, what are its dimensions $a$ and $b$?

9. A rectangle whose length is 8 feet greater than 3 times its width, contains 115 square feet. Find its dimensions, $a$ and $b$.

10. Find the base, $b$, and the altitude, $a$, of a rectangle whose area is 364 square miles and whose perimeter is 80 miles.

11. The longer side of a rectangle lacks 10 inches of being twice as long as the shorter side. The diagonal is 8 inches shorter than this double length. How long is the shorter side?

12. A rectangular building having a perimeter of 100 feet is inclosed by a fence of wire screen whose distance from the building is everywhere the same. The fence incloses an area 51 square feet larger than that occupied by the building. How far is the fence from the building?

13. A rectangular building having a perimeter of $2a$ feet is inclosed by a fence whose distance from the building is everywhere the same. The fence incloses an area $s$ square feet larger than that occupied by the building. How far is the fence from the building?

14. If from each of the four corners of a square sheet of tin, a 2-in. square is cut, and the four rectangular strips thus made are turned up into a square-cornered box of 50 cubic inches volume, what were the dimensions of the original square?

15. The area of a rectangle exceeds that of a square by 25 square inches. The side of the square is $\frac{5}{6}$ of the shorter side of the rectangle, and the longer side exceeds the shorter

side by 8 inches. Find the dimensions, $x$ and $y$, of the rectangle.

16. Solve Problem 15, using $a$ square inches instead of 25 square inches, $\frac{c}{d}$ instead of $\frac{5}{6}$, and $b$ inches instead of 8 inches.

17. The difference between the areas of a square and of a rectangle having the side of the square for one side and 20 units for the other, is 100 square units. Find the side of the square.

18. Solve a problem like 17, using $t$ in place of 20 and $s$ in place of 100.

19. One side of a rectangle is 10 units and the other is 2 units longer than the side of a square. The area of the rectangle exceeds that of the square by 80 square units. Find the side and area of the square.

20. Solve a problem like 19, replacing 10 by $p$, 2 by $q$, and 80 by $r$.

21. One side of a rectangle is 6 units and the other side is as long as the side of a square $\frac{3}{4}$ of whose area is $11\frac{1}{4}$ square units less than the area of the rectangle. Find the side and area of the square. Query: Is there more than one solution?

22. A rectangle having one side equal to the side of a square, and the other side $4\frac{1}{2}$ units less, exceeds $\frac{1}{2}$ the area of the square by 7 square units. Find the side and area of the square.

23. In Problem 21 replace 6 by $a$, $\frac{3}{4}$ by $\frac{b}{c}$, and $11\frac{1}{2}$ by $d$, and solve the problem thus made.

24. In Problem 22 replace $4\frac{1}{2}$ by $r$, $\frac{1}{2}$ by $\frac{1}{a}$, and 7 by $c$, and solve the problem thus formed.

25. The longer side of a rectangle is 99 centimeters shorter, and the diagonal is 27 centimeters shorter than double the shorter side. Find the sides and the diagonal of the rectangle.

26. The width of rectangular field is 20 rods less than the length. If a path connecting the opposite corners is 80 rods long, find the dimensions of the field.

27. The sides of a rectangle are 4 inches and 6 inches. By drawing lines parallel to two adjacent sides at equal distances from them, cut off $\frac{1}{4}$ of the given rectangle. How far from the sides must these lines be drawn?

28. Given two rectangles whose sides are $a$, $b$, and $c$, $d$ respectively. Compute the dimensions of. and construct a third rectangle, whose perimeter shall equal that of the first, and whose area shall equal that of the second given rectangle.

## § 14. Of Right Triangles

1. One leg of a right triangle is 7 feet longer than the other, and the hypothenuse is 14 feet longer than the shorter leg. Find the three sides.

2. The sides of a right triangle are as $3:4:5$. If the altitude upon the hypothenuse is 12 feet, what is the area?

3. What is the area of a right triangle of base 32, if the sum of the hypothenuse and the perpendicular to the base is 50?

4. In a triangle $AB$ is 9, $BC$ is 12, and $AC$ is 15. Into what sects does the bisector of angle $B$ divide $AC$?

5. A leg of a right triangle is 3 inches. Its projection upon the hypothenuse is 8 inches shorter than the hypothenuse. Find the hypothenuse.

6. A leg of a right triangle is $a$ inches, and its projection on the hypothenuse is $a$ inches shorter than the hypothenuse. Find the hypothenuse.

7. One leg of a right triangle is 8. The perpendicular from the vertex of the right angle divides the hypothenuse into sects, one of which is 8 less than 4 times the other. Find the hypothenuse.

8. One leg of a right triangle is $a$. The perpendicular

from the vertex of the right angle divides the hypothenuse into sects, one of which is $2a$ less than $a$ times the other. Find the hypothenuse.

9. In a right triangle the longer leg is 1 foot shorter than the hypothenuse and 7 feet longer than the short side. Find the length of each of the 3 sides.

10. In a right triangle the longer leg is $a$ feet shorter than the hypothenuse and $b$ feet longer than the short leg. Find the length of each of the three sides.

11. Find, to two places of decimals, each of the three sides of a right triangle whose longer leg is 1 yard shorter than the hypothenuse and 17 yards longer than the short leg.

12. A perpendicular from the vertex of the right angle of a right triangle upon the hypothenuse divides the hypothenuse so that the sect adjacent to the longer leg equals the shorter leg. How long is the shorter leg, if the hypothenuse is 1 foot long? Answer to two decimal places.

13. Solve a problem like 12 if the sect adjacent the longer leg equals twice the shorter leg and the hypothenuse is 10. Answer to two decimal places.

14. Solve a problem like 12, if the sect adjacent the longer leg is less than twice the shorter leg, and the hypothenuse is $3a$.

15. The hypothenuse of a right triangle is 8 units longer than the shorter leg, and the longer leg is 12 units long. How long is the hypothenuse?

16. In the middle of a pool 10 feet square grew a reed, which when it stood vertical, projected 1 foot above the surface of the water and, when blown aside, the top just reached to the middle of the side of the pool. How deep was the water? (Ancient Chinese problem.)

17. A perpendicular from the vertex of the right angle of a right triangle upon the hypothenuse is 12 inches long. The hypothenuse is 26 inches. Calling the legs $x$ and $y$, find them.

18. The sect, $y$, of the hypothenuse made by the perpendicular from the vertex of the right angle, is 12 inches shorter than the leg of the right triangle to which it is adjacent, and the other leg is 40 inches longer than the sect adjacent to it. Find the legs of the triangle and the sects of the hypothenuse.

19. Two points move uniformly from the vertex of a right angle along the sides. One starts 7 seconds later than the other, and 12 seconds after the first starts the points are 65 inches apart on a direct line. What are the rates of motion of the points?

20. Answer a problem like 19, replacing 7 by $a$, 12 by $b$, and 65 by $c$.

21. Two points lying, one on each of the sides of a right angle, begin moving toward the vertex at an instant when they are 13 feet apart. One moves 1 foot in $6\frac{2}{3}$ second and the other moves 1 foot in 1 minute. After 1 minute the points are 5 feet apart. How far were they from the vertex at the start?

22. Solve a problem like 21, replacing 13 by $a$, "1 foot in $6\frac{2}{3}$ seconds" by "1 foot in $b$ seconds," "1 foot in 1 minute" by "1 foot in $c$ minutes," and "after 1 minute the points are 5 feet apart" by "after $c$ minutes the points are $f$ feet apart."

23. Denoting by $a$, $b$, $c$, $p$, $q$, $h$, the lines beside which these symbols stand in Fig. 28, and by $A$, the area of the triangle, show that, if any two of them are given, the rest may be expressed in terms of the given two.

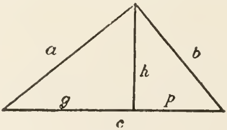

FIG. 28

24. The difference between the hypothenuse and one side of a right triangle is 4 feet. The other side is 8 feet long. Find the length, $l$, of the hypothenuse.

## § 15. Of Equilateral and Isosceles Triangles

### EQUILATERAL TRIANGLES

1. The altitude of an equilateral triangle is 42 feet. Find its side and its area.

2. The altitude of an equilateral triangle is $\frac{a}{2}\sqrt{3}$. Find its side and area.

3. The altitude of an equilateral triangle is $a$ feet. Find its side and area.

4. If the side of an equilateral triangle is 52 feet, what are the altitude and area?

5. What are the altitude and area of an equilateral triangle whose side is $a$ feet?

6. What are the altitude and area of an equilateral triangle whose side is $6a-14$ feet.

7. What are the altitude and side of an equilateral triangle whose area is $\frac{16}{3}\sqrt{3}$?

8. What are the altitude and side of an equilateral triangle whose area is $\frac{(a-6)^2}{2}$?

9. Let $c$ denote the side, $h$ the altitude, and $F$ the area of an equilateral triangle. Express (1) $c$ in terms of $h$ and $F$; (2) $h$ in terms of $c$ and $F$; (3) $F$ in terms of $c$ and $h$.

10. The side of an inscribed equilateral triangle is $a$. Compute the radius of the circumscribed circle.

11. The side of a circumscribed equilateral triangle is $a$. Compute the radius of the inscribed circle.

12. Given the radius of a circle, $r$. Express the side of the inscribed equilateral triangle in terms of $r$.

13. Express in terms of the radius, $r$, the side of a circumscribed equilateral triangle.

14. The sides $x$ and $y$ of two equilateral triangles are as $1 : 3$, and the sum of the two perimeters is 36. Find $x$ and $y$.

15. The sides, $x$ and $y$, of two equilateral triangles are as $3 : 8$, and the sum of the two perimeters is 99. Find $x$ and $y$.

16. The sides, $x$ and $y$, of two equilateral triangles are as $a : b$, and the sum of the two perimeters is $15(a+b)$.

17. The areas of two equilateral triangles are as $\frac{1}{16}$, and the sum of their perimeters is 45. Calling a side of one triangle $x$, and of the other, $y$; find $x$ and $y$.

18. The areas of two equilateral triangles are as $a : b$, and the sum of their perimeters is $6(a+b)$. Find the sides, $x$ and $y$, of the two triangles.

19. The areas of two equilateral triangles are as $1 : 4$ and the difference of their perimeters is 18. Find the sides, $x$ and $y$, of the two triangles.

20. The areas of two equilateral triangles are as $a : b$, and the difference of their perimeters is $3(a-b)$. Find the sides, $x$ and $y$, of the triangles.

21. Find the side, $a$, of an equilateral triangle whose area is 27 inches.

### Isosceles Triangles

1. A straight wire 30 inches long is bent into an isosceles triangle. Show that one side is less than 15 and another is greater than $7\frac{1}{2}$.

2. Given any two of the five, $c$, $a$, $h$, $w$, and $A$ (=area, see Fig. 29); to express any one of the others.

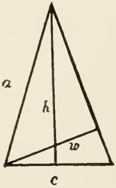

FIG. 29

3. Find the length of each leg of an isosceles right triangle, if the hypothenuse is 7 feet; 26 feet; $a$ feet; $a+1$ feet; $m+n$ feet.

4. Find the area of an isosceles triangle if the base is $b$ feet and one of the equal sides is $m$ feet.

5. The sides of an isosceles triangle are $x$, $x$, and $y$. Give its area in terms of $x$ and $y$.

6. The area of an isosceles triangle is 4 and one of the two equal sides is 5. Find all sides of the triangle.

7. Compute the sides of an inscribed square in terms of the radius, $r$, of the circle.

8. Given the radius, $r$, of a circle. Compute the side of a regular inscribed dodecagon.

9. Given the side, $s$, of a regular inscribed dodecagon. Compute the radius of the circle.

10. The area of a rhombus is 48 and one of its sides is 10. Find the diagonals.

11. One side of a rhombus is 10 inches, and one diagonal is 16 inches. Find the other diagonal and the area of the rhombus.

12. The base of one of two similar isosceles triangles is 2 and one of the equal sides is 3. The corresponding sides in the other triangle are $x$ and $y$. Find $x$ and $y$.

13. The two equal sides of one of two similar isosceles triangles are 3 and 3, and the sides of the other triangle are $x$, $x$, and $y$. The sum of the two perimeters is 48 and the ratio of the area of the first triangle to that of the second is $1:25$. Find all sides of both triangles.

14. Solve a problem like 13, if the sum of the perimeters is $s$, and the ratio of areas is $1:a^2$; other data remaining as in Problem 13.

15. Given an isosceles triangle whose vertex-angle is obtuse, whose base is $c$, and one of whose equal sides is $a$. Compute the length of a line which shall be parallel to the base and be a mean proportional between the side, $a$, and the upper sect of $a$, made by the parallel. Draw this parallel.

## § 16. Of Scalene Triangles

1. A straight wire 30 inches long was bent into a scalene triangle. Show algebraically that one side must have been less than 15, and the sum of the other two greater than 15.

2. The altitude of a triangle is 2 inches shorter than the base. The area is 140 square inches. Find the length, $x$, of the base and the altitude.

3. Let $a$, $b$, and $c$ denote the number of degrees in the angles of a triangle. The difference between $b$ and $a$ equals the difference between $c$ and $b$. One-half of $a$ is equal to one-third of $a$, added to one-fifth of $c$. Find the number of degrees in each angle.

4. The sum of one angle of a triangle, two times the second, and three times the third is 5 right angles. Four times the first, less three times the second, less the third, is also 5 right angles. Find the angles of the triangle.

5. The sums of the sides of a triangle, taken two at a time, are 8, 10, and 12, Find the length of each side.

6. The sides of a triangle are, $a=3$, $b=5$, and $AB=c=4$. Determine a position for $D$ along $AB$, Fig. 30, so that the line $CD$ divides the triangle into two triangles, having equal perimeters.

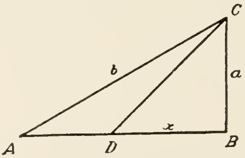

Fig. 30

7. A triangle whose base is 6 inches, and altitude, 9 inches is equal in area to an isosceles right triangle. Find the length of the side of the isosceles triangle.

8. If $s=\frac{1}{2}(a+b+c)$, show that (1) $a+b-c=2(s-c)$; (2) $a+c-b=2(s-b)$; and (3) $b+c-a=2(s-a)$.

9. If in the triangles of Fig. 31, $p$ denotes the projection of side $b$ upon side $c$, show that in the first triangle $p=\dfrac{b^2+c^2-a^2}{2c}$; and that in the second triangle $p=-\dfrac{b^2+c^2-a^2}{2c}$.

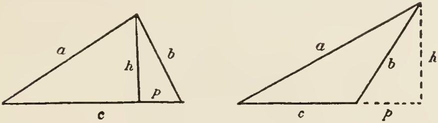

FIG. 31

10. Examine the forms of the values of $p$ in Problem 9 and write, by analogy, the corresponding formulas for the projections of $a$ upon $b$; for $a$ upon $c$; for $c$ upon $b$; for $b$ upon $a$; for $c$ upon $a$.

11. Noticing in Fig. 31 that $h^2=b^2-p^2$, show that in both cases,

$$h=\frac{1}{2c}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}.$$

12. By means of the results of Problem 2, show that the result of Problem 11 becomes—

$$h=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

13. Noting that the result of Problem 12 gives the altitude upon the side $c$, write, by analogy, the corresponding expressions for (1) the altitude upon side $a$; (2) the altitude upon side $b$.

14. The base of a triangle is $c$ and the altitude upon $c$ is $h$. Express the area, $A$, of the triangle.

15. Show from Problems 12 and 14 that the area, $A$, of any triangle in terms of its sides is given by—

$$A=\sqrt{s(s-a)(s-b)(s-c)}\,.$$

16. Show that the same expression is obtained for the area, $A$, of the triangle by using the first result of Problem 13; by using the second result of Problem 13.

17. Show that if $r$ denote the radius of a circle inscribed in a triangle of sides $a$, $b$, and $c$, the area, $A$, is given by $A=\frac{1}{2}r(a+b+c)$. (See Fig. 32.)

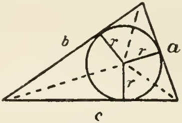

Fig. 32

18. Comparing the results of Problems 15 and 17, show that

$$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\,.$$

19. Find the radius of a circle that is inscribed in a triangle whose sides are 8, 10, and 15.

20. Find the area of the triangle whose sides are 8, 10, and 15.

21. The area of a triangle is 60 and the difference of the base and altitude is 8. Find the base, $x$, and altitude, $y$, of the triangle.

22. The area of a triangle is $\frac{1}{2}ab$, and the difference of the base and altitude is $b-a$. Find its base, $x$, and altitude, $y$.

23. The area of a triangle is 9, and 3 times the base exceeds the altitude by 3. Find the base and the altitude of the triangle.

24. The area of a triangle is $\frac{a^2}{2}$ and $a$ times the base exceeds the altitude by $a$. Find the base and altitude.

25. In any triangle of sides $a$, $b$, and $c$, show that (1) $c^2=a^2+b^2-2px$; and (2) $c^2=a^2+b^2-2qx$. ($p$ is the projection of $a$ on $b$, and $q$ is the projection of $a$ on $b$ produced.)

26. Given the three sides $a$, $b$, and $c$, of a scalene triangle. Express the sects, $x$ and $y$, of the base made by a perpendicular from the opposite vertex.

27. Find the sects of the side 15 of Problem 20 made by the perpendicular upon it from the opposite vertex.

28. In the triangle, $ABC$, side $AB$ is 9, side $BC$ is 12, and side $AC$ is 18. Into what sects, $x$ and $y$, does the bisector of the angle $B$ divide the side $AC$?

29. The sides of a triangle are $a$, $b$, and $c$. Find the sects $x$ and $y$, into which the bisector of the angle opposite side $c$, divides side $c$.

30. Write the sects, $x$ and $y$, into which side $b$ of the triangle of Problem 29 would be divided by the bisector of the angle opposite side $b$.

31. Write the sects, $x$ and $y$, of side $a$, made by the bisector of the opposite angle.

32. $BD$ is the bisector of angle $B$ in triangle $ABC$. The sects of $AC$ are $AD=7$ inches, and $DC=3$ inches. The sum of $AB$ and $BC$ is 16 inches. Find $AB$ and $BC$.

33. In triangle $ABC$, the side $a=10$ inches, $b=9$ inches, and $c=4$ inches. Find the parts of side $b$ that are made by the bisector of angle $B$.

34. Using the notation and suggestions of Fig. 33, prove that the sum of the squares of two sides of any triangle is equal to twice the square of half the third side increased by twice the square of the median upon that side.

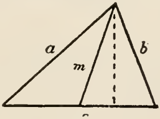

FIG. 33

35. Using the same figure (Fig. 33), prove that the difference of the squares of two sides of any triangle is equal to twice the product of the third side by the projection of the median upon that side.

36. From a figure like Fig. 34 express in terms of the sides $a$, $b$, and $c$ of the triangle, the radius, $r$, of the circle circumscribed about the triangle.

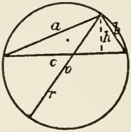

FIG. 34

37. Express the area of a triangle in terms of its sides $a$, $b$, and $c$, and the radius, $R$, of the circumscribed circle.

38. Prove from Fig. 35 that in any triangle the product of two sides equals the square of the bisector of the included angle, plus the product of the sects of the third side formed by the bisector (△BCE∼△DCA).

NOTE.—$h$ is not needed for Problem 38.

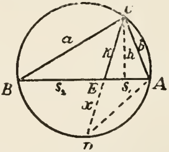

FIG. 35

39. From a figure like Fig. 35, but with $k+x$ passing through the center of the circle, prove that in any triangle the product of two sides is equal to the diameter of the circumscribed circle multiplied by the altitude upon the third side.

40. Find the length of a line from the vertex of a triangle to any point in the base, given the sides, $a$, $b$, $c$, and the sects, $m$ and $n$, of the base made by the point.

41. In a triangle having the sides 14 and 17 and the projection 4 of 14 upon 17, find the third side. (Use equations of Problem 26.)

42. The square of one side of a triangle is 1396. Another side is 28 and its projection upon the third side is 3. Find the third side. (See Problem 26.)

43. Given the sides $a$, $b$, and $c$ of any triangle, calculate the medians.

44. Given the sides $a$, $b$, and $c$ of any triangle, compute the lengths of the angle-bisectors.

45. Two triangles have a common angle. In one triangle the sides including this angle are 10 and 18 and the area is 90. In the other, the sides are $x$ and $y$ and the area is 60. Find $x$ and $y$, if $x+y=23$.

46. A straight line crosses the sides $a=25$, and $b=21$, of a triangle cutting off $\frac{1}{5}$ of the area of the triangle, and leaving a quadrilateral containing $\frac{4}{5}$ of the area of the triangle. The perimeter of the triangle that is cut away is 32 and the length of the cross-line is 10. Find the sects $x$ and $y$, that are cut off.

47. Divide a triangle into two equal parts by a line cutting across two of its sides, $a$ and $b$, and parallel to the third side, $c$.

48. Calling $x$ and $y$ the sects of $a$ and $b$ (Problem 47) that are adjacent the vertex of the included angle, find $x$ and construct it geometrically.

49. Divide a triangle into three equal parts by two parallels to a side, $c$, cutting across the other two sides, $a$ and $b$.

50. By a parallel to one side, $c$, of a triangle cutting the other two sides, $a$ and $b$, divide a triangle into two parts whose areas are to each other as $1:n$.

51. Calling $x$ and $y$ the upper sects of the sides $a$ and $b$ respectively, in Problem 50, show how to construct $x$ geometrically.

52. Divide the area and the perimeter of a triangle whose sides are 7, 8, and 9 into two equal parts by a line cutting across the sides 8 and 9.

53. Divide the area and perimeter of a triangle into two equal parts by a line cutting across its two sides $a$ and $b$.

54. The two values of $x$ in the last problem are $\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - \frac{ac}{2}}$ show how to construct geometrically the radical and the values of $x$.

55. Divide the area and perimeter of a triangle whose sides are 8, 9, and $c$ into two equal parts by a line cutting across the sides 8 and 9.

56. Divide the perimeter and the area of a triangle into thirds by a pair of parallel lines cutting across two of the sides.

57. Draw a line across two sides of a triangle cutting off $\frac{1}{n}$th of the perimeter and inclosing with these two sides $\frac{1}{n}$ of the area.

58. Draw a line as in Problem 51, cutting away $\frac{m}{n}$ths of the perimeter and area of the triangle.

59. The altitude of a triangle equals the side of a square and its base equals 9 units. One-seventh of the area of the triangle exceeds $\frac{1}{14}$ of the area of the square by 1 square unit. Find the side of the square and the altitude of the triangle.

60. If $a$, $b$, and $c$ denote the sides and $m$ the length of the median to the side $c$ of a triangle, prove that—

$$m = \tfrac{1}{2}\sqrt{2(a^2+b^2)-c^2}\,;$$

and write by analogy the expressions for the medians upon $a$; and upon $b$.

61. The median of a triangle is 5 inches long and the side to which it is drawn is 10 inches long. The bisector of the angle from whose vertex the median is drawn divides the opposite side into parts that are as 1 : 2. Find the unknown sides, $x$ and $y$, of the triangle.

62. After finding $x$ and $y$ of Problem 61, prove the triangle to be a right triangle.

63. An angle-bisector and a median, drawn from the same vertex of a triangle, intercept on the opposite side, which is 16 inches long, a sect 2 inches long. The median is 7 inches long. Find the two unknown sides, $x$ and $y$.

64. Solve a problem like 63, supposing the opposite side to be 21, the sect to be 5 and the median to be $m$ in length.

65. A median and an angle-bisector from the same vertex are respectively 4 and 5 inches long, and the opposite side is 10 inches long. Find the unknown sides, $x$ and $y$.

66. Solve a problem like 65, taking the median $m$ inches long, the bisector $b$ inches, and the opposite side 21 inches long.

67. Find the altitude, $h$, of a triangle whose sides are 7, 9, and 12 inches long. (See Problem 6.)

68. Prove that the length, $k$, of the bisector of the angle $C$ of the triangle whose sides are $a$, $b$, and $c$ (angle $C$ opposite side $c$), is given by

$$k=\frac{2}{a+b}\sqrt{a\,b\,s(s-c)}\,.$$

69. In the triangle of Problem 68, write by analogy the expression for the length of the bisector of angle $A$ opposite side $a$; of angle $B$.

70. Using the notation of Fig. 36, and supposing $a$ to denote the area of the triangle, show the following three equations to

be true, if the perpendicular is so drawn as to bisect the area of the triangle:

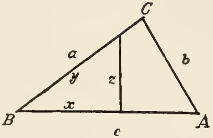

FIG. 36

(1) $xy=\frac{1}{2}ac$; (2) $xz=\frac{1}{2}A$; and (3) $x^2+z^2=y^2$.

71. Show how to divide a triangle whose sides are $a$, $b$, and $c$ into two equal parts by drawing a perpendicular to $c$.

72. Write the corresponding formulas for bisecting the area of a triangle by a perpendicular to side $c$, *when $b$ is longer than $a$*.

73. Write the corresponding formulas for a perpendicular to $a$, bisecting the area of the triangle whose sides are $a$, $b$, and $c$.

74. Write the formulas for a perpendicular to $b$, bisecting the area.

75. Find $x$, $y$, and $z$ of Problem 64, if $a=4$, $c=7\frac{1}{2}$, and $A=24$.

76. Show that equation (2) of Problem 64 might be replaced by $xz=\frac{1}{2}hc$, $h$ denoting the altitude upon $c$.

## § 17. Of Similar Figures

1. The sides of a triangle are 8, 10, and 13. The shortest side of a similar triangle is 11. Find the other sides.

2. The sides of a triangle are 1, 2, and 3, and the longest side of a similar triangle is 20. Find the rest of the sides of the second triangle.

3. The sides of a triangle are 56, 42, and 7 inches. Find the sides of a similar triangle whose perimeter is 26 inches.

4. In the accompanying figure (Fig. 37) A′B′ is parallel to AB. If $AC=12$, $AB=20$, $BC=16$, and $A'C=3$, find $x$ and $y$.

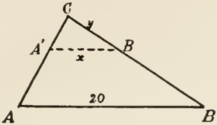

FIG. 37

5. The side of one square is 5 times as long as that of another square, and its area is 72 square yards greater than that of the second square. How long is the side of the first square?

6. The sides of a triangle are $a=3$, $b=4$, and $c=5$. A line $a'$, whose length is 2 is drawn within the triangle parallel to $a$, dividing the triangle into two parts. What part of the whole triangle is the upper portion?

7. Suppose $AC$ (Fig. 37) were 32 and $BC$ were 36, how long must $A'C$ and $B'C$ be in order that the line $A'B'$, parallel to $AB$, divide the triangle into two equal parts?

8. Write the equations for dividing a triangle of sides $a$, $b$, and $c$, into two equal parts (1) by a parallel to side $C$; (2) by a parallel to $b$; and (3) by a parallel to $c$.

9. Show by the use of algebraic numbers to denote lengths of lines that a perpendicular to the hypothenuse of a right triangle from the vertex of the right angle divides the hypothenuse (1) so that each side about the right angle is a mean proportional between the whole hypothenuse and its adjacent segment; (2) so that the segments of the hypothenuse are to each other as the squares of the adjacent sides of the right triangle; (3) so that the perpendicular itself is a mean proportional between the sects of the hypothenuse.

10. The sum of the areas of two squares is 164 and the

sides are to each other as 4 : 5. Find the sides of the two squares.

11. The difference of the areas of two squares is 80 and the sides are to each other as 3 : 2. Find the lengths of the sides of the two squares.

2 The sum of the areas of two similar rectangles is 120, and the areas of two similar rectangles is 120, and the areas are to each other as 1 : 4. Find the lengths of the sides oi both rectangles.

13. The areas of two regular pentagons are to each other as 4 : 25, and a side of one of the pentagons is 3 units longer than that of the other. Find the sides of the pentagons.

14. Two watermelons of the same shape have corresponding diameters in the ratio of 2 : 3. The price of the larger is 25c. and of the smaller 15c. Which contains the larger quantity of edible matter for 1 cent, if the thickness of the rinds is proportional to the corresponding diameters?

## § 18. Of Parallelograms and Trapezoids

### PARALLELOGRAMS

1. The area of a rhombus is 48 and one of its sides is 10. Find the diagonals, $2x$ and $2y$, of the rhombus.

2. One side of a rhombus is 10 inches, and one diagonal 16 inches. Find the other diagonal, $d$, and the area, $A$, of the rhombus.

3. The perimeter of a parallelogram is 80 and one side is 4 times as long as the adjacent side. Find the sides, $x$ and $y$, of the parallelogram.

4. Find the lengths of the two altitudes of the parallelogram of Problem 3, if the area is 640.

5. Two adjacent sides of a parallelogram are $5x+8$ and $3x-5$ and the perimeter is 166. Find the lengths of the sides.

6. One side of a parallelogram is 12 and the perimeter is 84. Find the remaining sides.

7. Using the notation and relations suggested by Fig. 38, prove, algebraically, that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

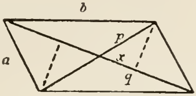

Fig. 38

8. Find the base, $b$, and the altitude, $a$, of a parallelogram whose area is 16 and whose base is 6 units longer than its altitude.

9. Find the base, $y$, and altitude, $x$, of a parallelogram whose area is $2ab+b^2$, (1) the base being $a$ greater than the altitude; and (2) the base being $a$ less than the altitude.

10. Find the base and the altitude of a paralellogram, whose area is 135 and whose base is 12 less than 3 times the altitude.

11. Find the base and the altitude of a parallelogram whose area is $2n+1$ and whose base is (1) $4n$ greater than 4 times the altitude, and (2) $4n$ less than 4 times the altitude.

12. The base of a parallelogram is $x+8$, the altitude is. $x-8$, and the area is 80. Find $x$, the base, and the altitude.

13. The altitude of a parallelogram is 18 less than the base and the number of square units in the area is 56 greater than 8 times the number of linear units in the base. Find the base.

14. The dimensions of one of two parallelograms having equal areas are $x$ and $x+6a$, and of the other, $a$ and $375a-4x$. Find $x$.

### Trapezoids

1. The difference of the two parallel sides of a trapezoid is 4, the altitude of the trapezoid is 2 and its area is 32. Find the lengths of the parallel sides.

2. The difference between the longer one of the parallel sides of a trapezoid and the double of the other is 5, the altitude is 6, and the area is 168. Find the parallel sides, $x$ and $y$.

3. The difference of two parallel sides of a trapezoid is $d$, its altitude is $a$, and its area is $A$. Find the lengths of the parallel sides.

4. The line joining the middle points of the non-parallel sides of a trapezoid is 13 feet and 5 inches, and one of the parallel sides is $2\frac{1}{2}$ times the other. Find the lengths, $x$ and $y$, of the parallel sides.

5. If one base of a trapezoid is 12.8 inches and the lines joining the mid-points of the non-parallel sides is 3.2 times the other base, find the other base.

6. The area of a trapezoid is $\frac{h}{2}(a+b)$, and the difference of the parallel sides, $x$ and $y$, is $a-b$. Find the values of $x$ and $y$.

7. The line connecting the mid-points of the non-parallel sides of a circumscribed trapezoid equals $\frac{1}{4}$ of the sum of the sides $a$, $b$, $c$, and $d$ of the trapezoid. Prove algebraically.

8. The bases of a trapezoid are 15 and 28 and its altitude is 5. If the non-parallel sides are produced until they meet, find the area of the triangle formed with the shorter base.

9. The area of a trapezoid is $\frac{hx+hy}{2}$. One base equals twice the other, and the altitude equals $\frac{2}{3}$ of the shorter base. Find the altitude and bases of the trapezoid.

10. The area of a trapezoid is $45(a+b)$ and the altitude is 9. What is the length of the line connecting the mid-points of the non-parallel sides?

11. The parallel sides of a trapezoid are 15 and 10 and the altitude is 7. Find the altitudes of the similar triangles produced by extending the non-parallel sides until they meet.

12. The area of a trapezoid is 36, the altitude, $2x$, and the parallel bases are $9-17x$ and $23x-12$. Find $x$.

## § 19. Of Regular Polygons and Circles

### REGULAR POLYGONS

1. If $a$ denotes the apothem of a regular inscribed polygon, $s$, the side and $r$, the radius of the circle, show that—

(1) In a regular inscribed triangle, $s = r\sqrt{3}$; $a = \frac{1}{2}r$;
(2) In an inscribed square, $s = r\sqrt{2}$; $a = \frac{1}{2}r\sqrt{2}$;
(3) In a regular inscribed hexagon, $s = r$; $a = \frac{1}{2}r\sqrt{3}$.

2. The apothem of a regular pentagon is 6 and a side is 4. Find the perimeter and area of a regular pentagon whose apothem is 8.

3. The sum of the perimeters of two regular pentagons is 100, and their areas are as 1 : 9. Calling a side of one pentagon $x$, and the other $y$, find $x$ and $y$.

4. In Problem 3, replace the first part of the statement by, "the difference of the perimeters is 50." Find $x$ and $y$.

5. Compute the side of an inscribed equilateral triangle whose radius is 6 inches.

6. Compute the side of an inscribed equilateral triangle whose radius is $r$ inches. Express the result as a radical.

7. Compute the side, $s$, and area, $A$, of a regular circumscribed hexagon, if the radius of the circle is 7.

8. Compute the side, $s$, and the area $A$, of a regular circumscribed hexagon, the radius of the circle being $r$. Express the results in radicals.

9. Compute the radius of a regular inscribed triangle whose radius is $r$.

10. Compare the areas of the regular inscribed hexagon and the regular inscribed triangle, the radius of the circle in each case being $r$.

11. The side of a regular hexagon is 60. Find its area.

### CIRCLES

1. In a circle a chord 80 feet long is 30 feet from the center. Find the radius, $r$.

2. If the radii of two intersecting circles are $r_1$, and $r_2$, and the sects of the line of centers made by the common chord are $s_1$, and $s_2$, prove algebraically, that—

$$(1)\ r_2^2-r_1^2=s_2^2-s_1^2, \quad \text{and} \quad (2)\ \frac{r_2+r_1}{s_2+s_1}=\frac{r_2-r_1}{s_2-s_1}.$$

3. Find the sects $s_1$, and $s_2$, if $r_1=8$, $r_2=6$, and $s_1+s_2=12$.

4. A diameter of a circle is divided into parts 4 inches and 9 inches by a chord that is perpendicular to it. Find the length of the chord.

5. Inscribe a square within the common part of two intersecting circles.

Suggestion:

$$\left.\begin{array}{l} \cos \alpha=\dfrac{s}{R}; \\[2ex] \sin \alpha=\dfrac{R-a}{R}; \end{array}\right\} s^2-2aR+a^2=0;$$

$$\left.\begin{array}{l} \sin \beta=\dfrac{s}{r}; \\[2ex] \cos \beta=\dfrac{r-b}{r}; \end{array}\right\} s^2-2br+b^2=0, \quad \text{and } 2s=d-a-b.$$

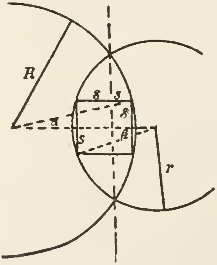

FIG. 39

6. What is the radius of a circle if it is 26 inches shorter than a chord of the same circle, which is 7 inches from the center?

7. What is the diameter of a circle if it is 4 inches longer than the chord that is distant $\frac{3}{10}$ of the diameter from the center?

8. Given the line $AB$, and $C$ its mid-point. On $AB$, $AC$, and $BC$ semicircles are drawn, all having their diameters coinciding with $AB$. Compute the radius, $x$, of a circle touching each of the three given semicircles.

9. Show how to draw the semicircle of radius, $x$, of Problem 8.

10. Express in terms of the radii, $r$ and $r'$, the common area of two intersecting circles.

11. What shape does the equation answering Problem 10 assume when the two radii become equal?

12. The diameter of a circle is divided into three equal parts and semicircles are drawn as shown in Fig. 40. Express the ratio of each part of the area of the original circle to the whole area of the original circle.

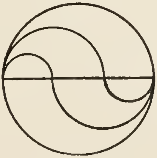

FIG. 40

## §20. Of Sets of Points and Lines

1. How many lines can be drawn through 4 points, no 3 of which are in the same straight line?

2. How many lines can be drawn through 5 points, no 3 of which lie in the same straight line? How many through 6 points? 7 points? 12 points? $n$ points?

3. How many points, no 3 in the same straight line, are there in a group in which just 55 connecting lines arise by connecting them in all possible ways?

4. How many points in a point-group giving, under the conditions of Problem 3, just 105 connecting lines? 300 connecting lines?

5. How many diagonals has a quadrilateral? A pentagon? A hexagon?

6. How many diagonals has a heptagon? An octagon? A nonagon? A decagon? An 11-gon? A 15-gon? An $n$-gon?

7. Two polygons have together 12 sides and 19 diagonals. How many sides has each?

8. Two polygons have together 14 sides and 29 diagonals. How many sides has each?

9. Two polygons have together 18 sides and the number of diagonals of the one is to the number of diagonals of the other as 4 : 7. How many sides has each?

10. A polygon of $n$ sides has 9 diagonals more than a polygon of $n-1$ sides. Find the number of sides of the first polygon.

11. If the numbers of sides of two polygons are consecutive numbers, how do the numbers of diagonals compare?

12. How many more diagonals are there in a polygon of 8 sides than in a polygon of 7 sides?

## CUBIC EQUATIONS

13. Show that the number, $t$, of triangles that arise by connecting $n$ points, 3 and 3, in all possible ways, is given by—

$$t=\frac{n(n-1)(n-2)}{3 \cdot 2}.$$

14. How points are there in a point-group in which 4 triangles arise by connecting the points, 3 and 3, in all possible ways?

15. How many points are there in a group which gives rise to 20 triangles by connecting the points, 3 and 3, in all possible ways? How many points, if 120 triangles arise?

16. The total number of sides of two polygons is 21 and the number of diagonals of one is to that of the other as 1 : 2. How many sides has each polygon?

## §21. Of Cubes and Parallelopipeds

### CUBES

1. The edge of a cube is $7\frac{3}{4}$ cm; find its surface, $s$, and its volume, $v$.

2. Find the length, $d$, of the diagonal of the cube of Problem 1.

3. The edge of a cube is $12\frac{1}{2}$. Find its surface, $s$, its volume, $v$, and its diagonal, $d$.

4. The edge of a cube is $a+b$. Find its diagonal, $d$, its surface, $s$, and its volume, $v$.

5. The edge of a cube is $3a-b$. Find its diagonal, $d$, its surface, $s$, and its volume, $v$.

6. The edge of a cube is $a+b+c$. Find its diagonal, $d$, its surface, $s$, and its volume, $v$.

7. What are the edge, $e$, the surface, $s$, and the diagonal, $d$, of a cube whose volume is $a^3+3a^2b+3ab^2+b^3$?

8. The volume of a cube is $64a^3-144a^2c+108ac^2-27c^3$. Find the edge, $e$, the surface, $s$, and the diagonal, $d$. Find also the length, $l$, of a face-diagonal.

9. The edge of one cube is 3 inches longer than double the edge, $x$, of another cube. The surface of the larger cube is 1350 square inches. Find the edge, volume, and diagonal of both cubes.

10. The edge of one cube is 8 inches longer than 3 times the edge, $x$, of another cube, and the volume of the first cube is 4096. Find the edge, the surface, and the diagonal of both cubes.

11. The volume of a cube is $421\frac{7}{8}$ and its edge is $5x-9$. Find $x$.

12. The number of square units in the surface of a cube equals the number of cubical units in its volume. Find its edge, its surface, and diagonal.

13. The edge of a cube is 5. Find the area, $a$, of the rectangle made by a diagonal plane, through opposite edges.

14. A plane through the edge, $AD$, cuts the opposite face in $EF$, at the distance, $x$, from the edge diagonally opposite to the edge $AD$. The edge of the cube is 8 and the volume of the wedge $AD-CBFE$, is 192. Find $x$, and denoting $DE$ by $y$, find $y$, and also the area $AFED$.

15. A plane like $AE$, Fig. 41, cuts a cube so that $x^2-16x+128=100$. Find $x$ and the area of the rectangle, $FBCE$, made by the cutting plane. What is the meaning of the two values of $x$ obtained?

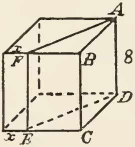

FIG. 41

16. An oblique plane (see Fig. 42) parallel to the vertical edges, cuts a cube whose edge is 8 as shown. Show that if $d=10$, $x$ is given by—

$$4x^2-32x+128=100.$$

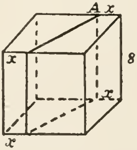

FIG. 42

Solve the equation for $x$, and interpret both results.

17. How long is the edge, $x$, of a cube, if the volume increases 19 cubic inches for an increase of 1 inch in the edge?

18. How long is the edge, $x$, of a cube for a volumetric increase of 81 cubic inches corresponding to an increase of 1 inch in the length of the edge?

### PARALLELOPIPEDS

1. The edges of a rectangular parallelopiped are 3 inches, 4 inches, and 5 inches, respectively. Find the area, $s$, of its surface; the volume, $v$, and the diagonal, $d$ (to two decimals).

2. The edges of a rectangular parallelopiped are 4 inches, 6 inches, and $8b$ inches, respectively. Find the area, $s$, of the surface, the volume, $v$, and the length, $d$, of the diagonal.

3. The edges of a rectangular parallelopiped are $x$, $y$, and $z$. Express the surface-area, $s$, the volume, $v$, and the square of the diagonal, $d^2$, in terms of $x$, $y$, and $z$.

4. Find the edges, $x$, $y$, and $z$, of a rectangular parallelopiped, if the area of the face of dimensions $x$ and $y$ is 12, and the edge $z$ is 3 longer than edge $x$.

5. The sum of the 3 edges, $x$, $y$, and $z$, of a rectangular parallelopiped is 9, the volume is $\frac{1}{9}$ of the volume of a cube whose edge is 6, and 3 times the edge $x$ exceeds 10 by the length of edge $z$. Find all the edges.

6. The sum of the edges, $x$, $y$, and $z$, of a rectangular parallelopiped is 6, If the face area of dimensions $x$ and $y$ is 2, and that of dimensions $y$ and $z$ is 6, find the edges $x$, $y$, and $z$.

7. The base and the altitude of a parallelogram that forms the base of a parallelopiped are $x$ and $y$ respectively, and the altitude of the parallelopiped is $z$. The sum of the dimensions is 10, the area of the base is 15, and the altitude of the base is to the altitude of the parallelopiped as $3:2$. Find $x$, $y$, and $z$.

8. The first, second, and third dimensions of a rectangular

parallelopiped are $x$, $y$, and $z$, respectively. The sum of twice the first, 3 times the second, and 4 times the third is 20; of 3 times the first, 4 times the second and 5 times the third is 26; of 3 times the first, 5 times the second, and 6 times the third is 31. Find the dimensions, the volume, the surface, and the diagonal of the parallelopiped.

9. $x$, $y$, and $z$ denoting the first, second, and third dimensions of a rectangular parallelopiped, find the dimensions, the volume, and the surface-area, if the sum of $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{5}$ of the third is 23; of $\frac{1}{3}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{2}$ of the third is 27; and of $\frac{1}{5}$ of the first, $\frac{1}{6}$ of the second, and $\frac{1}{3}$ of the third is 17.

10. $x$, $y$, and $z$ as before, find them, the volume, and the surface-area, if the sum of $\frac{1}{2}$ the first, and $\frac{1}{3}$ the second exceeds 12 by $\frac{1}{6}$ of the third; the sum of $\frac{1}{2}$ of the second and $\frac{1}{3}$ of the third falls short of 8 by $\frac{1}{6}$ of the first; and the sum of $\frac{1}{2}$ the first and $\frac{1}{3}$ of the third is 10.

11. $x$, $y$, and $z$ as before, find them, the volume, and the surface-area, if the sum of the reciprocals of the first and the second is $\frac{1}{4}$; of the second and third is $\frac{7}{36}$, and of the first and the third is $\frac{5}{18}$.

12. $x$, $y$, and $z$ as before, find them, the volume, and the surface-area, if the sum of the reciprocals of the first and second is $a$, of the second and third is $b$, and of the first and third is $c$.

13. By increasing each edge of a rectangular parallelopiped 1 unit the volume was increased 46 cubic units, the surface 50 square units, and the area of the face of dimensions $x$ and $y$, 6 square units. Find the edges $x$, $y$, and $z$, and the volume of the original parallelopiped.

## §22. Of Prisms and Pyramids

### PRISMS

1. The base of a right prism is $b$ square feet and the length of a lateral edge $a$ feet. Find $a$ and $b$ if the volume is 216

cubic feet and there are 6 times as many square feet in the area of the base as linear feet in the edge.

2. Find the volume of a regular hexagonal prism, the perimeter of whose base is 28 and whose altitude equals a side of the base.

3. One side of the square base of a rectangular prism of altitude $3x$, is $2x$. Find $x$ if the volume is 1,500 cubic feet.

4. The dimensions of a right prism, whose base is a rectangle 6 by 8, are 6, 8, and 15. Find the total surface.

5. The total area of a right prism, whose base is a rectangle, is 118, the volume is 70, and the altitude is 7. Find the dimensions of the base.

6. Find the volume of a right prism whose base is a rectangle, whose surface is .262, and the dimensions of whose base are 7 and 8.

7. If the side of the base of a regular hexagonal prism is 6, and the altitude is 15, find the lateral area and the volume.

8. The base of a right prism is a right triangle whose sides are 9 and 15. The volume is 270. Find the lateral area.

9. A right triangular prism has basal sides 5, 6, and 7, and the altitude of the prism is 8. Find the lateral area and the volume of the prism.

10. Find the lateral area of a regular triangular prism whose volume is $72\sqrt{3}$ and one side of whose base is 6.

11. The base of a regular triangular prism is $25\sqrt{3}$. The altitude is $\sqrt{75}$. Find the volume, and the lateral area.

12. Find the volume of a prism whose altitude is 5 inches and whose base is a parallelogram one of whose diagonals is 8 inches and whose adjacent sides are 6 inches and 7 inches.

13. The base of a right prism is a trapezoid whose parallel sides are 6 feet and 8 feet. One of the non-parallel sides is 4 feet and the diagonal is 7 feet. The altitude of the prism is 16 feet. Find the volume of the prism.

14. What is the volume of a prism with an altitude of 10

inches, whose base is a parallelogram having diagonals of 6 inches and 8 inches, and a side of 5 inches?

15. The base of a right triangular prism is 126 square feet. Two of its sides are 20 feet and 21 feet. Find the lateral area.

16. The lateral area and volume of a regular prism whose base is a hexagon are 252 and $378\sqrt{3}$, respectively. What is the altitude and a side of the base?

17. One side of the bottom of a square cistern is $x$, and the cistern is $x+6$ feet deep. Find the side of the base and the depth, if the cistern holds $160x$ cubic feet.

18. The cross-section of a bridge timber, 24 feet long, is a regular octagon of 8-inch sides. How many cubic feet of timber does it contain? Answer (1) with a radical, and (2) with a decimal to two places.

## PYRAMIDS

1. The edge of a regular tetrahedron is 8 inches; find the volume, the lateral surface, and the total surface of the tetrahedron.

2. Express the lateral surface, the total surface, and the volume of a regular tetrahedron whose edge is $e$ inches long.

3. The base of a regular pyramid is a square of side 12 and the lateral faces are equilateral triangles. Find the volume, $v$, the lateral surface, $l$, and the total surface, $t$, to three decimal figures.

4. Express the volume, $v$, the lateral surface, $l$, and the total surface, $t$, in terms of the edge, $e$, of a pyramid such as is described in Problem 3.

5. Solve a problem like 3, assuming the side of the base to be 12 and each lateral edge to be 16.

6. Solve a problem like 5, supposing the sides of the base to be $s$, and each lateral edge to be $e$.

7. Supposing the base of a regular pyramid to be a square

of side $(a+b)$ and the altitude of the pyramid to be $(a+b)$; find the lateral edge, $l$, and the total area of the surface in terms of $a$ and $b$.

8. Find the edge, $e$, the lateral area, $l$, and the total area, $t$, of a regular triangular pyramid whose volume is $576\sqrt{2}$ cubic inches.

9. Find the edge, $e$, the lateral area, $l$, the total area, $t$, of a regular triangular pyramid whose volume is $9\sqrt{2}(2a-b)^3$.

10. A pyramid with an equilateral triangle of side $a$ for its base and with lateral edges all equal to $b$, has a volume of $\frac{a^3}{12}$ cubic inches. Find the relation between $a$ and $b$.

11. Any geometry will give further exercises on pyramids, that may be readily converted into algebraic exercises if desired.

## §23. Of Cylinders and Cones

### CYLINDERS

1. What is the lateral area, $A$, the total area, $T$, and the volume, $V$, of a right circular cylinder whose diameter is 6 inches and whose altitude is 8 inches?

2. What is the lateral area, $A$, the total area, $T$, and the volume, $V$, of the right circular cylinder whose altitude is $h$, and the circumference of whose base is $c$?

3. Denoting the radius of a right cylinder by $r$, the lateral area by $A$, the total area by $T$, and the volume by $V$, and calling the ratio of the circumference of a circle to its diameter $\pi$, derive the following relations:

$$(1)\ rh=\frac{A}{2\pi};\qquad (2)\ r^2+hr=\frac{T}{2\pi};\qquad \text{and } (3)\ r^2h=\frac{V}{\pi}.$$

4. With letters denoting the same as in Problem 3, derive the following formulas:

$$(1)\ r=\frac{V}{A};\qquad (2)\ r^2+hr=\frac{T+A}{2\pi};$$

(3) $r^2=\frac{T-A}{2\pi}$; and (4) $\frac{h}{r}=\frac{T-A}{A}$.

5. Using for $\pi$, $3\frac{1}{7}$, and supposing $h$ to equal 20 and $T$ to be $109\frac{1}{7}$, find the radius, $r$, the lateral area, $A$, and the volume, $V$, of a right circular cylinder.

6. Compare the lateral area, $A$, of a right circular cylinder of radius, $r$, with the lateral area, $L$, of the regular inscribed hexagonal prism.

7. Compare the lateral area, $A$, of a right circular cylinder of radius, $r$, with the lateral area, $L'$, of the regular circumscribed hexagonal prism.

8. The volume of a right circular cylinder is equal to the volume of a 6-inch cube, and the lateral area of the cylinder equals the total surface of a 3-inch cube. Find the radius, $r$, and altitude, $h$, of the cylinder, using $\pi=3\frac{1}{7}$.

9. Compare the volume, $V$, the lateral surface, $L$, and the total surface, $T$, of a right circular cylinder of radius, $r$, and altitude, $2r$, with the volume, $V'$, the lateral surface, $L'$, and the total surface, $T'$, respectively, of a right circular cone of the same base and altitude.

## CONES

1. The radius of the base of a right circular cone is 6 and the altitude is 9. Find the lateral area, $A$, the total area, $T$, and the volume, $V$.

2. Express the lateral area, $A$, the total area, $T$, and the volume, $V$, of a right circular cone in terms of the radius of the base and the altitude.

3. Letting $l$ denote the slant height, $r$ the radius, and $h$ the altitude of a right circular cone show the following to be true relations:

(1) $r^2=(l-h)(l+h)$; and (2) $h^2=(l-r)(l+r)$.

4. What are the lateral area, $A$, the total area, $T$, and the

volume, $V$, of the frustum of a cone of revolution, the radii of whose bases are 4 and 12 and whose altitude is 16?

5. What is the volume of the frustum of a cone whose altitude is 14 feet and the circumferences of whose bases are 18 feet and 24 feet.

6. The volumes of two similar cones of revolution are as 343 to 1,000. What is the ratio of their lateral areas? Of their total areas? Of their altitudes? Of the radii of their bases?

7. Compare the volumes of a cone of revolution of radius, $r$, and altitude, $h$, and a cylinder of revolution of radius, $r$, and altitude, $h$.

8. Compare the lateral areas of the two figures of revolution of Problem 7.

9. The area of the lateral surface of a right circular cone is $16\pi$, and the total surface is $80\pi$. Find the radius, $r$, the slant height, $l$, and the altitude, $h$, of the cone.

10. Denoting the radius of a right circular cone by $r$, the altitude by $h$, the slant height by $l$, the lateral area by $A$, the total area by $T$, and the volume by $V$, derive the following relations

$$(1)\ rl=\frac{A}{\pi}; \qquad (3)\ r^2h=\frac{3V}{\pi};$$

$$(2)\ r^2+rl=\frac{T}{\pi}; \qquad \text{and } (4)\ r^2=\frac{T-A}{\pi}.$$

11. From the relations of Problem 10 express $l$, $r$, and $h$ in terms of $A$, $T$, $V$, and $\pi$.

12. The difference of the total and lateral areas of a right circular cone is $25\pi$ and the volume is $75\pi$. Find the radius, $r$, the altitude, $h$, and the slant height, $l$.

13. Letting $r$, $r'$, $V$, and $h$ denote respectively, the radii of the lower and of the upper bases, the volume and the altitude of a cone of revolution, show that—

$$r^2+2rr'+r'^2=\frac{3V}{\pi h}+rr' .$$

14. Given $V=80\pi$, $h=12$, and $rr'=16$, in the frustum of a right circular cone. Find the radii, $r$ and $r'$, of the bases and the lateral area, $A$, of the frustum.

## § 24. Of Spheres

1. Find the circumference, $c$, the surface, $s$, and the volume, $v$, of a sphere whose diameter is 9 inches.

2. Express the circumference, $c$, the surface, $s$, and the volume, $v$, of a sphere whose radius is $r$.

3. The diameter of a base-ball is 1.75 inches. Find its surface, $s$, and its volume, $v$.

4. The area of a spherical zone is $1200\pi$, and the altitude of the zone is 10 greater than the radius. Find the radius, $r$, of the sphere and altitude, $h$, of the zone.

5. If $h$, $r$, $r'$, and $V$ denote the altitude, the radii of the lower and upper bases, and the volume of a spherical segment, show that—

$$\frac{6V}{\pi}=3h(r^2+r'^2)+h^3 .$$

6. If the volume of a spherical sector is $600\pi$, show that if $r$ and $h$ denote its radius and altitude, respectively,

$$r^2h=900 .$$

7. Show that if the volume of a spherical sector is $V$ and the radius of the sphere is $r$ and the altitude of the segment is $h$,

$$r^2h=\frac{3V}{\pi} .$$

8. In a spherical segment $V=167\frac{20}{21}$, and $h=10$. Calling $\pi=3\frac{1}{7}$, find $r$ and $r'$ if their difference, $r-r'$, is 5.

9. Compare the volume and surface of a sphere of radius,

$r$, with the volume and surface, respectively, of an inscribed cube.

10. Compare the volume and surface of a sphere of radius $r$, with the volume and surface, respectively, of a circumscribed cube.

11. Compare the volume, $V$, and surface, $S$, of a sphere of radius, $r$, with the volume, $V'$, and total surface, $S'$, of the circumscribed right cylinder.

12. Compare the volume, $V$, and surface, $S$, of a sphere of radius, $r$, with the volume, $V'$, and the total surface, $S'$, of a right circular cone of radius, $r$, and altitude, $2r$.

13. Any geometry will furnish further numerical or literal exercises on the Sphere, which may readily be put into form for algebraic solution.

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